This book presents original mathematical models of thermal stresses in composite materials, along with mathematical models of thermal-stress induced micro-/macro-strengthening and thermal-stress induced intercrystalline or transcrystalline crack formation. The mathematical determination results from mechanics of an isotropic elastic continuum. The materials consist of an isotropic matrix with isotropic ellipsoidal inclusions. The thermal stresses are a consequence of different thermal expansion coefficients of the matrix and ellipsoidal inclusions.
The mathematical models include microstructural parameters of a real matrix-inclusion composite, and are applicable to composites with ellipsoidal inclusions of different morphology (e.g., dual-phase steel, martensitic steel). In case of a real matrix-inclusion composite, such numerical values of the microstructural parameters can be determined, which result in maximum values of the micro- and macro-strengthening, and which define limit states with respect to the intercrystalline or transcrystalline crack formation in the matrix and the ellipsoidal inclusion. This numerical determination is performed by a programming language.

Dr. Ladislav Ceniga, DSc. (Institute of Materials Research, Slovak Academy of Sciences, Kosice, Slovak Republic) works on mathematical models of stresses in composites.

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Thermal Stresses in Composites with Ellipsoidal Components

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This book was reviewed by the following reviewers:

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Táto kniha je venovaná s láskou mojim najdrahším rodičom a starým rodičom.

This book is dedicated with love to my dearest parents and grandparents.

Contents

Introduction

This book[1] presents original mathematical models of thermal stresses in composite materials, along with mathematical models of thermal-stress induced micro-/macro-strengthening and thermal-stress induced intercrystalline or transcrystalline crack formation. The materials consist of an isotropic matrix with isotropic ellipsoidal inclusions. These stresses originate during a cooling process, and are a consequence of different thermal expansion coefficients of the matrix and ellipsoidal inclusions.

The mathematical models are determined for a suitable model system. The model system is required to correspond to real isotropic matrix-inclusion composites. The thermal stresses are derived within a suitable coordinate system. The coordinate system is required to correspond to a shape of the ellipsoidal inclusions.

The mathematical determination results from mechanics of an isotropic elastic continuum, and result in different mathematical solutions for the thermal stresses, i.e., 19 and 2 mathematical solutions for the matrix and the ellipsoidal inclusion, respectively. Due to these different mathematical solutions, the principle of minimum elastic energy is considered.

The mathematical models of the thermal stresses and of the thermal-stress induced micro-/macro-strengthening and crack formation include microstructural parameters of a real matrix-inclusion composite, i.e., the inclusion dimensions a_1, a_2, a_3, the inclusion volume fraction v_{IN}, as well as the inter-inclusion distance $d = d\left(a_1, a_2, a_3, v_{IN}\right)$.

Consequently, the mathematical models are applicable to composites with ellipsoidal inclusions of different morphology, i.e., $a_1 \approx a_2 \approx a_3$ (dual-phase steel), $a_1 \gg a_2 \approx a_3$ (martensitic steel).

In case of a real matrix-inclusion composite, such numerical values of the mi-

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Assoc. Prof. Ing. Daniel Kottfer, PhD., Alexander Dubček University of Trenčín, Faculty of Special Technology Department of Mechanical Engineering, Trenčín, Slovak Republic crostructural parameters can be determined,

- which result in maximum values of the micro- and macro-strengthening,
- which define limit states with respect to the intercrystalline or transcrystalline crack formation in the matrix and the ellipsoidal inclusion.

This numerical determination is performed by a programming language. The mathematical procedures in this book are analysed in Appendix.

Chapter 1

Matrix-Inclusion Composite

1.1 Model System

Figure 1.1 shows a model system, corresponding to real matrix-inclusion composites, which is considered within the mathematical models of the thermal stresses. This model system consists of an infinite isotropic matrix and isotropic ellipsoidal inclusions with the dimensions a_1, a_2, a_3 and the inter-inclusion distance d along the axes x_1, x_2, x_3 of the Cartesian system $(Ox_1x_2x_3)$, respectively, where O represents a centre of the ellipsoidal inclusion.

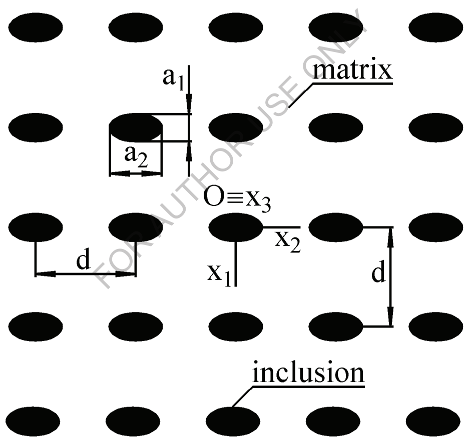


Figure 1.1: The matrix-inclusion system with an infinite isotropic matrix and isotropic ellipsoidal inclusions with the dimensions a_1, a_2, a_3 and the inter-inclusion distance d along the axes x_1, x_2, x_3 of the Cartesian system $(Ox_1x_2x_3)$, respectively, where O represents a centre of the ellipsoidal inclusion.

As presented in [1]–[22], the thermal stresses are determined in the cubic cells with the dimension d along the axes x_1, x_2, x_3 and with central ellipsoidal inclusions (see Figure 1.2). Due to the infinite matrix, the thermal stresses, which are determined for one of the cubic cells, are identical with those, which are determined for any of the cubic cells [1]–[22]. With regard to the volume $V_{IN} = 4\pi a_1, a_2 a_3$ [23] and $V_C = d^3$ of the ellipsoidal inclusion and the cubic cell, the inter-inclusion distance d as a function of the inclusion volume fraction v_{IN} is derived as

$$v_{IN} = \frac{V_{IN}}{V_C} = \frac{4\pi a_1 a_2 a_3}{3d^3} \in \left(0, \frac{\pi}{6}\right), \quad d = \left(\frac{4\pi a_1 a_2 a_3}{3v_{IN}}\right)^{1/3}, \tag{1.1}$$

where the value $v_{INmax} = \pi/6$ results from the condition $a_i \to d/2$ $(i=1,2,3)$. Accordingly, the thermal stresses are functions of the material parameters a_1, a_2, a_3, v_{IN}, d.

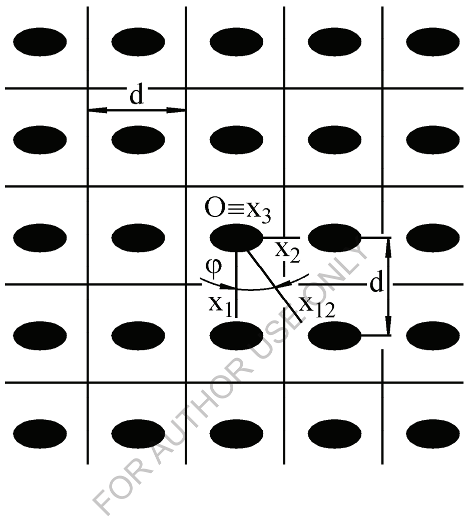


Figure 1.2: The cubic cells with the dimension d along the axes x_1, x_2, x_3 of the Cartesian system $(Ox_1x_2x_3)$ and with the plane $x_{12}x_3$, where O represents a centre of the ellipsoidal inclusion, and ($x_{12} \subset x_1x_2$, $x_{12}x_3 \perp x_1x_2$.

1.2 Coordinate System

Figure 1.3 shows the ellipse E with the dimensiions a, b along the axes x, y, respectively. The ellipse E is described by the function

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1. \tag{1.2}$$

Any point P of the ellipse E is described by the coordinates [23]

$$x = a\cos\alpha, \quad y = b\sin\alpha, \quad \alpha \in \langle 0, 2\pi \rangle, \tag{1.3}$$

where the normal n of the ellipse E at the point P is derived [23]

$$\frac{\partial x}{\partial \alpha}(x - a\cos\alpha) + \frac{\partial y}{\partial \alpha}(x - b\sin\alpha) = 0. \tag{1.4}$$

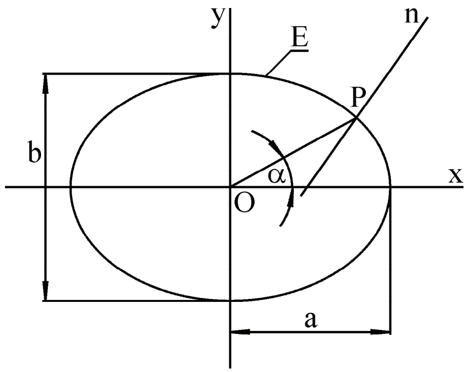


Figure 1.3: The ellipse E with the dimensions a, b along the axes x, y of the Cartesian system (Oxy), respectively, and the point P related to the angle α.

With regard to Equations (1.3), (1.4), we get

$$y = \frac{x\,a\tan\alpha}{b} - \frac{\left(a^2 - b^2\right)\sin\alpha}{b}. \tag{1.5}$$

The thermal stresses are determined by the spherical coordinates $(r, \varphi\nu)$ (see Figure 1.4). The model system in Figures (1.1), (1.2) is symmetric, and then the thermal stresses are determined within the intervals $\varphi \in \langle 0, \pi/2\rangle$, $\nu \in \langle 0, \pi/2\rangle$ [1]–[22].

Figure 1.4 shows the ellipsoidal inclusion for $\varphi, \nu \in \langle 0, \pi/2\rangle$ with the centre O and with the dimensions $a_1 = O1$, $a_2 = O2$, $a_3 = O3$ along the axes x_1, x_2, x_3 of the Cartesian system $(O{,}x_1{,}x_2{,}x_3)$ (see Figures (1.1), (1.2)), respectively. With regard to Equation (1.3), any point of the ellipse E_{12} in the plane x_1x_2 is described by the coordinates

$$x_1 = a_1\cos\varphi, \quad x_2 = a_2\sin\varphi, \quad \varphi \in \left\langle 0, \frac{\pi}{2}\right\rangle. \tag{1.6}$$

Similarly, any point P of the ellipse E_{123} in the plane $x_{12}x_3$ is described by the coordinates

$$x_{12P} = a_{12}\sin\nu, \quad x_{3P} = a_3\cos\nu, \quad a_{12} = O4 = \sqrt{a_1^2\cos^2\varphi + a_2^2\sin^2\varphi},$$
$$\varphi, \nu \in \left\langle 0, \frac{\pi}{2}\right\rangle. \tag{1.7}$$

Finally, $(P, x_n, x_\varphi, x_\nu)$ is a Cartesian system at the point P, where the axes x_n and x_ν represents a normal and a tangent of the ellipse E_{123} at the point P, respectively, $x_{12}x_3 \perp x_1x_2$, $(x_{12} \subset x_1x_2$, $x_\varphi \perp x_{12}$.

Figure 1.5 shows the cross section $O567$ of the cubic cell in the plane $x_{12}x_3$ (see Figures 1.2, 1.4). The angle $\nu \in \langle 0, \pi/2 \rangle$ defines a position of the point P with the Cartesian system $(P, x_n, x_\varphi, x_\nu)$ (see Figure 1.4) for $\nu = \nu_0$ (see Figure 1.5a), $\nu \in \langle 0, \nu_0)$ (see Figure 1.5b), $\nu \in (\nu_0, \pi/2\rangle$ (see Figure 1.5c). The points P_1, P_2 represent intersections of the normal x_n with $O567$.

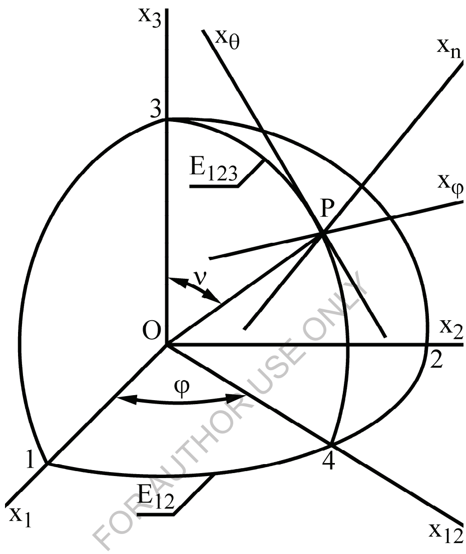


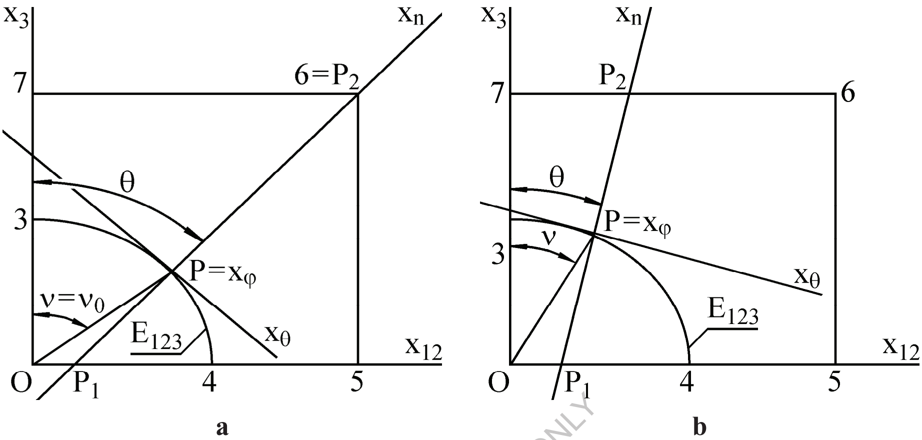
Figure 1.4: The inclusion with the centre O and with the dimensions $a_1 = O1$, $a_2 = O2$, $a_3 = O3$ along the axes x_1, x_2, x_3 of the Cartesian system (O, x_1, x_2, x_3), respectively. The ellipses E_{12}, E_{123} in the planes x_1x_2, $x_{12}x_3$ (see Figure 1.4) are given by Equations (1.6), (1.7), respectively, where $x_{12}x_3 \perp x_1x_2$, $(x_{12} \subset x_1x_2$, $x_\varphi \perp x_{12}$. The point P on the inclusion surface is defined by $\varphi, \nu \in \langle 0, \pi/2 \rangle$, $\nu \in \langle 0, \pi/2 \rangle$, and $(P, x_n, x_\varphi, x_\nu)$ is a Cartesian system at the point P, where $P \subset E_{123}$. The axes x_n and x_ν represents a normal and a tangent of the ellipse E_{123} at the point P, respectively.

With regard to Equation (1.5), the normal x_n at the point P of the ellipse E_{123} in the plane $x_{12}x_3$ is derived as

$$x_3 = \frac{\cos\nu}{a_3}\left(\frac{a_{12}x_{12}}{\sin\nu} + a_3^2 - a_{12}^2\right), \quad \nu \in \left\langle 0, \frac{\pi}{2} \right\rangle. \tag{1.8}$$

With regard to Equation (1.8), the coordinates $x_{x_{12,1}}$, $x_{3,1}$ of the point P_1 have the forms

$$x_{12\,1} = \frac{\left(a_{12}^2 - a_3^2\right)\sin\nu}{a_{12}}, \quad x_{3\,1} = 0, \quad \nu \in \left\langle 0, \frac{\pi}{2} \right\rangle. \tag{1.9}$$



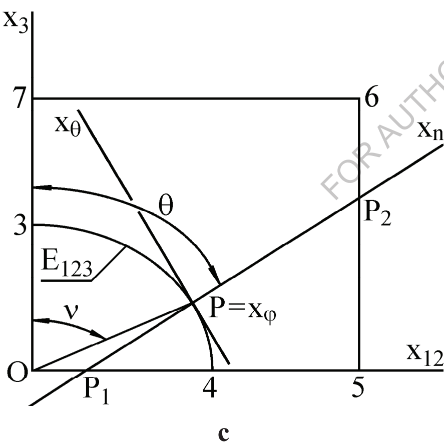


Figure 1.5: The angle $\nu \in \langle 0, \pi/2 \rangle$ defines a position of the point P with the Cartesian system $\left(P, x_n, x_\varphi, x_\nu\right)$ (see Figure 1.4) for (a) $\nu = \nu_0$, (b) $\nu \in \langle 0, \nu_0 \rangle$, (c) $\nu \in (\nu_0, \pi/2\rangle$, where ν_0 is given by Equation (1.8). The points P_1, P_2 represent intersections of the normal x_n with $O567$, where $O567$ is a cross section of the cubic cell in the plane $x_{12}x_3$ (see Figures 1.2, 1.4). The angle $\theta\angle\left(x_n, x_3\right)$ is given by Equation (1.12).

Similarly, the coordinates $x_{x_{12,2}}$, $x_{3,2}$ of the point P_2 in Figure 1.5b for $\nu \in \langle 0, \nu_0 \rangle$ are derived as

$$x_{12\,2} = \frac{\sin\nu}{a_{12}} \left(\frac{d\cos\nu}{2\,a_3} + a_{12}^2 - a_3^2 \right), \quad x_{3\,2} = \frac{d}{2}, \quad \nu \in \langle 0, \nu_0 \rangle. \tag{1.10}$$

The coordinates $x_{x_{12,2}}$, $x_{3,2}$ of the point P_2 in Figure 1.5c for $\nu \in \langle \nu_0, \pi/2 \rangle$ have the forms

$$
\begin{aligned}
&x_{12\,2} = \frac{d}{2\,f(\varphi)\sin\nu}, \\
&f(\varphi) = \cos\varphi, \quad \varphi \in \left\langle 0, \frac{\pi}{4} \right\rangle; \quad f(\varphi) = \sin\varphi, \quad \varphi \in \left\langle \frac{\pi}{4}, \frac{\pi}{2} \right\rangle, \\
&x_{3\,2} = \frac{\cos\nu}{a_3}\left[\frac{a_{12}\,d}{2\,f(\varphi)\sin\nu} + a_3^2 - a_{12}^2\right], \quad \nu \in \left\langle \nu_0, \frac{\pi}{2} \right\rangle.
\end{aligned}
\tag{1.11}
$$

The coordinate $x_{12,2}$ of the point P_2 in Figure 1.5a for $\nu = \nu_0$ is given by Equation (1.11), where $x_{3\,2} = d/2$. With regard to Equation (1.8), the angle ν_0 represents a root of the following equation

$$
\begin{aligned}
&\frac{\cos\nu_0}{a_3}\left[\frac{a_{12}\,d}{2\,f(\varphi)\sin\nu_0} + a_3^2 - a_{12}^2\right] - \frac{d}{2} = 0, \\
&f(\varphi) = \cos\varphi, \quad \varphi \in \left\langle 0, \frac{\pi}{4} \right\rangle; \quad f(\varphi) = \sin\varphi, \quad \varphi \in \left\langle \frac{\pi}{4}, \frac{\pi}{2} \right\rangle,
\end{aligned}
\tag{1.12}
$$

and this root is determined by a numerical method. The angle $\theta = \angle(x_n, x_3)$ is derived as

$$
\begin{aligned}
&\cos\theta = \frac{x_{3P}}{\sqrt{(x_{12P} - x_{12\,1})^2 + x_{3P}^2}} = \frac{1}{\sqrt{1 + (a_3 \tan\nu / a_{12})^2}}, \\
&\sin\theta = \frac{1}{\sqrt{1 + (a_{12} \cot\nu / a_3)^2}}.
\end{aligned}
\tag{1.13}
$$

Consequently, we get [23]

$$
\frac{\partial}{\partial\theta} = \left(\frac{\partial\theta}{\partial\varphi}\right)^{-1} \frac{\partial}{\partial\varphi} = \Theta \frac{\partial}{\partial\varphi},
\tag{1.14}
$$

where the function $\Theta = \Theta(\varphi)$ has the form

$$
\Theta = \left(\frac{a_{12}}{a_3}\right)\left[\left(\frac{a_3 \sin\nu}{a_{12}}\right)^2 + \cos^2\nu\right].
\tag{1.15}
$$

As analysed in [1]-[20], due to the symmetry of the model system, any point P on the matrix-inclusion boundary exhibits the displacement u_n along x_n. Consequently, any point P of the normal x_n exhibits u_n along x_n, i.e., $u_\varphi = u_\nu = 0$ [1]-[20], where u_φ, u_ν are displacements along the axes x_φ, x_ν, respectively.

As presented in [1]–[22], the thermal stresses, which are determined along the axes x_n, x_φ, x_θ of the Cartesian system $\left(P, x_n, x_\varphi, x_\theta\right)$, represent function of the spherical coordinates (x_n, φ, θ) for $\varphi, \theta \in \langle 0, \pi/2 \rangle$. The intervals $x_n \in \langle 0, x_{IN} \rangle$ and $x_n \in \langle x_{IN}, x_M \rangle$ are related to the ellipsoidal inclusion and the cell matrix, where $P = P_1$, $P \subset E_{123}$ and $P = P_2$ for $x_n = 0$, $x_n = x_{IN}$ and $x_n = x_M$ (see Figure 1.5), respectively. Finally, we get

$$x_{IN} = P_1 P = \sqrt{\left(x_{12P} - x_{121}\right)^2 + x_{3P}^2} = a_3 \sqrt{\left(\frac{a_3 \sin\nu}{a_{12}}\right)^2 + \cos^2\nu}, \tag{1.16}$$

$$\begin{aligned} x_M = PP_2 &= \sqrt{\left(x_{122} - x_{12P}\right)^2 + \left(x_{32} - x_{3P}\right)^2} \\ &= \sqrt{\left(\frac{\sin\nu}{a_{12}}\right)^2 \left(\frac{d \cos\nu}{2 a_3} - a_3^2\right)^2 + \left(\frac{a_{12} \cos\nu}{a_3}\right)^2 \left[\frac{d}{2 f(\varphi) \sin\nu} - a_{12}\right]^2}. \end{aligned} \tag{1.17}$$

Chapter 2

Mechanics of Elastic Solid Continuum

2.1 Fundamental Equations

As analysed in [1]-[20], any point P of the normal x_n exhibits the displacement u_n along x_n. The thermal stresses are determined along the axes x_n, x_φ, x_θ of the Cartesian system $\left(P, x_n, x_\varphi, x_\theta\right)$. Fundamental equations of mechanics of a solid continuum are represented by Cauchy's equations, the equilibrium equations and Hooke's law. Cauchy's equations represent functions of strains and displacements. With respect to the normal displacement u_n, Cauchy's equations have the forms [1]-[20, 22]

$$\varepsilon_n = \frac{\partial u_n}{\partial x_n}, \tag{2.1}$$

$$\varepsilon_\varphi = \varepsilon_\theta = \frac{u_n}{x_n}, \tag{2.2}$$

$$\varepsilon_{n\varphi} = \varepsilon_{\varphi n} = \frac{1}{x_n}\,\frac{\partial u_n}{\partial \varphi}, \tag{2.3}$$

$$\varepsilon_{n\theta} = \varepsilon_{\theta n} = \frac{\Theta}{x_n}\,\frac{\partial u_n}{\partial \nu}, \tag{2.4}$$

where ε_n is a normal strain along the axis x_n, and Θ is given by Equation (1.15). Consequently, ε_φ and ε_θ are tangential strains along the axes x_φ and x_θ, respectively. Finally, $\varepsilon_{n\varphi}$, $\varepsilon_{n\theta}$ and $\varepsilon_{\varphi n}$, $\varepsilon_{\theta n}$ represent shear strains along the axes x_n and x_φ, x_θ, respectively. Due to $u_\varphi = u_\nu = 0$, we get $\varepsilon_{\varphi\nu} = \varepsilon_{\nu\varphi} = 0$ [1]–[22], where u_φ, u_ν are displacements along the axes x_φ, x_ν, respectively, and $\varepsilon_{\varphi\nu}$ is a shear strain. As presented in [1]–[22], the equilibrium equations are derived as

$$2\sigma_n - \sigma_\varphi - \sigma_\nu + x_n \frac{\partial \sigma_n}{\partial x_n} + \frac{\partial \sigma_{n\varphi}}{\partial \varphi} + \Theta \frac{\partial \sigma_{n\theta}}{\partial \nu} = 0, \tag{2.5}$$

$$\frac{\partial \sigma_\varphi}{\partial \varphi} + 3\sigma_{n\varphi} + x_n \frac{\partial \sigma_{n\varphi}}{\partial x_n} = 0, \tag{2.6}$$

$$\Theta \frac{\partial \sigma_\theta}{\partial \nu} + 3\sigma_{n\theta} + x_n \frac{\partial \sigma_{n\theta}}{\partial x_n} = 0, \tag{2.7}$$

where σ_n is a normal stess along the axis x_n. Consequently, σ_φ and σ_θ are tangential stresses along the axes x_φ and x_θ, respectively. Finally, $\sigma_{n\varphi}$, $\sigma_{n\theta}$ and $\sigma_{\varphi n}$, $\sigma_{\theta n}$ represent shear stresses along the axes x_n and x_φ, x_θ, respectively, where $\sigma_{n\varphi} = \sigma_{\varphi n}$, $\sigma_{n\theta} = \sigma_{\theta n}$. Due to $\varepsilon_{\varphi\nu} = \varepsilon_{\nu\varphi} = 0$, we get $\sigma_{\varphi\nu} = \sigma_{\nu\varphi} = 0$ [1]–[22], where $\sigma_{\varphi\nu}$ is a shear stress. With regard to $\varepsilon_{\varphi\theta} = 0$, $\sigma_{\varphi\theta} = 0$, Hooke's law has the form [1]–[20, 22]

$$\varepsilon_n = s_{11}\sigma_n + s_{12}\left(\sigma_\varphi + \sigma_\theta\right), \tag{2.8}$$

$$\varepsilon_\varphi = s_{12}\left(\sigma_n + \sigma_\theta\right) + s_{11}\sigma_\varphi, \tag{2.9}$$

$$\varepsilon_\theta = s_{12}\left(\sigma_n + \sigma_\varphi\right) + s_{11}\sigma_\theta, \tag{2.10}$$

$$\varepsilon_{n\theta} = s_{44}\sigma_{n\theta}, \tag{2.11}$$

$$\varepsilon_{n\varphi} = s_{44}\sigma_{n\varphi}, \tag{2.12}$$

where s_{11}, s_{12}, s_{44} are derived as [24]

$$s_{11} = \frac{1}{E}, \quad s_{12} = -\frac{\mu}{E}, \quad s_{44} = \frac{2\left(1+\mu\right)}{E}. \tag{2.13}$$

Finally, E and μ are Young's modulus and Poisson's ratio, respectively. In case of the ellipsoidal inclusion and the cell matrix, we get $E = E_{IN}$, $\mu = \mu_{IN}$ and $E = E_M$, $\mu = \mu_M$, respectively. With regard to Equations (2.1)–(2.4), (2.8)–(2.12), we get [1]–[22]

$$\sigma_n = \left(c_1 + c_2\right)\frac{\partial u_n}{\partial x_n} - 2c_2\frac{u_n}{x_n}, \tag{2.14}$$

$$\sigma_\varphi = \sigma_\theta = -c_2\frac{\partial u_n}{\partial x_n} + c_1\frac{u_n}{x_n}, \tag{2.15}$$

$$\sigma_{n\varphi} = \frac{1}{s_{44}x_n}\frac{\partial u_n}{\partial \varphi}, \tag{2.16}$$

$$\sigma_{n\theta} = \frac{\Theta}{s_{44}x_n}\frac{\partial u_n}{\partial \nu}, \tag{2.17}$$

where c_1, c_2, c_3 (see Equation (2.24)) have the forms

$$c_1 = \frac{E}{(1+\mu)(1-2\mu)}, \quad c_2 = -\frac{\mu E}{(1+\mu)(1-2\mu)}, \quad c_3 = -4(1-\mu) < 0, \tag{2.18}$$

and $c_3 < 0$ due to $\mu < 0.5$ for real isotropic components [25].

Let $a_{1i} = \cos[\angle(x_1, x_i)]$ $(i = n, \varphi, \theta)$ represent a direction cosine of an angle formed by the axes x_1, x_i (see Figures 1.4, 1.5). With regard to Figures 1.4, 1.5, the coefficient $a_{1i} = \cos[\angle(x_1, x_i)]$ $(i = n, \varphi, \theta)$ is derived as

$$\begin{aligned} &a_{1n} = \cos\varphi \sin\theta, \quad a_{1\varphi} = \sin\varphi \sin\theta, \quad a_{1\theta} = \cos\theta, \\ &a_{\varphi 1} = -\sin\varphi, \quad a_{\theta 1} = -\cos\varphi \cos\theta, \end{aligned} \tag{2.19}$$

where $\cos\theta$, $\sin\theta$ are given by Equation (1.13). The stress σ_1 along the axis x_1 has the form

$$\sigma_1 = a_{1n}\sigma_n + a_{1\varphi}\sigma_\varphi + a_{1\theta}\sigma_\theta + a_{1n}\left(\sigma_{n\varphi} + \sigma_{n\theta}\right) + a_{1\varphi}\sigma_{\varphi n} + a_{1\theta}\sigma_{\theta n}. \tag{2.20}$$

With regard to Equations (2.14)-(2.17) and due to $\sigma_{n\varphi} = \sigma_{\varphi n}$, $\sigma_{n\theta} = \sigma_{\theta n}$ [24], we get

$$\sigma_1 = \gamma_1 \frac{\partial u_n}{\partial x_n} + \gamma_2 \frac{u_n}{x_n} + \frac{1}{s_{44} x_n}\left(\gamma_3 \frac{\partial u_n}{\partial \varphi} + \gamma_4 \frac{\partial u_n}{\partial \nu}\right), \tag{2.21}$$

where γ_i $(i = 1, \ldots, 4)$ is derived as

$$\begin{aligned} &\gamma_1 = a_{1n}(c_1 + c_2) - (a_{1\varphi} + a_{1\theta})\, c_2, \quad \gamma_2 = (a_{1\varphi} + a_{1\theta})\, c_1 - 2 a_{1n} c_2, \\ &\gamma_3 = a_{1n} + a_{1\varphi}, \quad \gamma_4 = \Theta\,(a_{1n} + a_{1\theta}), \end{aligned} \tag{2.22}$$

and Θ is given by Equation (1.15). As presented in Chapter 8, the analytical models of the micro-strengthening $\sigma_{st} = \sigma_{st}(x_1)$ and the macro-strengthening $\overline{\sigma_{st}}$ result from the stress σ_1 (see Equations (2.21), (2.22)).

Let Equations (2.14)–(2.17) be substituted to Equation (2.18) and to $[\partial \mathrm{Eq.}(2.6)/\partial\varphi]$ + $\Theta\,[\partial \mathrm{Eq.}(2.7)/\partial\nu]$. Consequently, Equations (2.5)–(2.7) are derived as

$$x_n^2 \frac{\partial^2 u_n}{\partial x_n^2} + 2 x_n \frac{\partial u_n}{\partial x_n} - 2 u_n + \frac{U_n}{s_{44}(c_1 + c_2)} = 0, \tag{2.23}$$

$$x_n \frac{\partial U_n}{\partial x_n} = c_3 U_n, \tag{2.24}$$

where U_n is derived as

$$U_n = \frac{\partial^2 u_n}{\partial \varphi^2} + \Theta^2 \frac{\partial^2 u_n}{\partial \nu^2}. \tag{2.25}$$

The system of the differential equations (2.23), (2.25) is solved by the mathematical procedures in Sections 3.1, 4.1, 5.1, 6.1, 7.1.

2.2 Elastic Energy

As analysed in [1]–[22] with respect to the different mathematical procedures (see Sections 3.1, 4.1, 5.1, 6.1, 7.1), such a mathematical solution, which exhibits a minimum value of the elastic energy W_C of the cubic cell, is considered, where W_{IN} and W_M is elastic energy, which is accumulated in the volume V_{IN} and V_M of the ellipsoidal inclusion and the cell matrix, respectively. The elastic energy density w is derived as [24]

$$w = \frac{1}{2}\left(\varepsilon_n \sigma_n + \varepsilon_\varphi \sigma_\varphi + \varepsilon_\theta \sigma_\theta\right) + \varepsilon_{n\varphi}\sigma_{n\varphi} + \varepsilon_{n\theta}\sigma_{n\theta}, \tag{2.26}$$

and W_{IN}, W_M and W_C have the forms

$$\begin{aligned} W_{IN} &= \int\limits_{V_{IN}} w_{IN}\, dV_{IN} = 8 \int\limits_0^{\pi/2} \int\limits_0^{\pi/2} \int\limits_0^{x_{IN}} w_{IN}\, x_n^2\, dx_n\, d\varphi\, d\nu, \\ W_M &= \int\limits_{V_M} w_M\, dV_M = 8 \int\limits_0^{\pi/2} \int\limits_0^{\pi/2} \int\limits_{x_{IN}}^{x_M} w_M\, x_n^2\, dx_n\, d\varphi\, d\nu, \\ W_C &= W_{IN} + W_M. \end{aligned} \tag{2.27}$$

2.3 Boundary Conditions

The mathematical solutions of the system of the differential equations (2.23), (2.25) include integration constants. As presented in [1]–[22], these constants are determined, using Cramer's rule (see Chapter 8) [23], by the following boundary conditions for the ellipsoidal inclusion and the cell matrix. In case of the ellipsoidal inclusion we get [1]–[22]

$$(u_n)_{x_n=0} = 0, \tag{2.28}$$

$$(\sigma_{nIN})_{x_n=x_{IN}} = -p_n, \tag{2.29}$$

where x_{IN} is given by Equation (1.16). Additionally, the conditions $(u_{nIN})_{x_n \to 0} \not\longrightarrow \pm\infty$, $(\varepsilon_{IN})_{x_n \longrightarrow 0} \not\longrightarrow \pm\infty$, $(\sigma_{IN})_{r \to 0} \not\longrightarrow \pm\infty$ are required to be fulfilled [1]–[22].

In case of the cell matrix we get [1]–[22]

$$(\sigma_{nM})_{x_n = x_{IN}} = -p_n, \tag{2.30}$$

$$(u_{nM})_{x_n = x_M} = 0. \tag{2.31}$$

As analysed in [1]–[22], the following boundary condition can be considered

$$(\varepsilon_{nM})_{x_n = x_M} = 0. \tag{2.32}$$

With regard to $\left(\varepsilon_{\varphi M}\right)_{x_n = x_M} = -p_n \rho_M$, $\left(\varepsilon_{\varphi IN}\right)_{x_n = x_{IN}} = -p_n \rho_{IN}$ [1]–[22], the normal stress p_n on the matrix-inclusion boundary, i.e., for $x_n = P_1P = x_{IN}$ (see Figure 1.5), which acts along the axis x_n (see Figures (1.4), (1.5)), has the form [1]–[22]

$$p_n = \frac{(\alpha_{IN} - \alpha_M)(T_r - T)}{\rho_M + \rho_{IN}}, \tag{2.33}$$

where $T_r = (0.35 - 0.4) \times T_m$ [25] and T_m is relaxation and melting temperature of a real composite system, respectively, and T is final temperature of a cooling process.

As mentioned in Section (2.2), the different mathematical procedures in Sections 3.1, 4.1, 5.1, 6.1, 7.1 result in 19 and 2 mathematical solutions for the thermal stresses in the matrix and the ellipsoidal inclusion, respectively.

The normal stress p_n is included in formulae for the thermal stresses. Consequently, the coefficients ρ_M and ρ_{IN} are given by Equations (3.29), (4.26), (4.37), (4.48), (4.59), (5.25), (5.36), (5.47), (5.58), (6.24), (6.35), (6.46), (6.57), (7.25), (7.36), (7.47), (7.58), (7.69), (7.80) and (3.37), (6.65), respectively. Consequently, such a combination of ρ_M and ρ_{IN} is considered to result in a minimum value of the elastic energy W_C of the cubic cell (see Equation (2.27)).

Chapter 3

Mathematical Model 1

3.1 Mathematical Procedure

Let the mathematical procedure $x_n \left[\partial \text{Eq.}(2.24)/\partial x_n\right]$ be performed, and then we get [1]–[22]

$$x_n^2 \frac{\partial^2 U_n}{\partial x_n^2} + (1 - c_3)\, x_n \frac{\partial U_n}{\partial x_n} = 0, \tag{3.1}$$

where $c_3 < 0$ and $U_n = U_n(x_n, \varphi, \theta)$ are given by Equations (2.18) and (2.25), respectively. Let Equation (2.24) be substituted to Equation (3.1), and then we get [1]–[22]

$$x_n^2 \frac{\partial^2 U_n}{\partial x_n^2} + c_3 (1 - c_3)\, U_n = 0. \tag{3.2}$$

Let U_n be assumed in the form $U_n = x_n^{\lambda}$, then we get [1]–[22]

$$U_n = C_1 x_n^{\lambda_1} + C_2 x_n^{\lambda_2}, \tag{3.3}$$

where C_1, C_2 are integration constants, which are determined by the boundary conditions in Section 2.3, and λ_1, λ_2, with respect to $\mu < 0.5$ for a real isotropic material [25], have the forms [1]–[22]

$$\begin{aligned} \lambda_1 &= \frac{1}{2}\left[1 + \sqrt{1 + 16(1-\mu)\left[1 + 4(1-\mu)\right]}\right] > 3, \\ \lambda_2 &= \frac{1}{2}\left[1 - \sqrt{1 + 16(1-\mu)\left[1 + 4(1-\mu)\right]}\right] < -2. \end{aligned} \tag{3.4}$$

Let Equation (3.3) be substituted to Equation (2.23), and then we get [1]–[22]

$$x_n^2 \frac{\partial^2 u_n}{\partial x_n^2} + 2 x_n \frac{\partial u_n}{\partial x_n} - 2 u_n = C_1 x_n^{\lambda_1} + C_2 x_n^{\lambda_2}. \tag{3.5}$$

The mathematical solution of Equation (3.5), which is determined by Wronskian's method (see Chapter 10) [23], is derived as

$$u_n = C_1 x_n^{\lambda_1} + C_2 x_n^{\lambda_2}. \tag{3.6}$$

With regard to Equations (2.1)–(2.4), (2.14)–(2.17), (2.21), (2.26), (3.6), we get

$$\varepsilon_n = C_1 \lambda_1 x_n^{\lambda_1 - 1} + C_2 \lambda_2 x_n^{\lambda_2 - 1}, \tag{3.7}$$

$$\varepsilon_\varphi = C_1 x_n^{\lambda_1 - 1} + C_2 x_n^{\lambda_2 - 1}, \tag{3.8}$$

$$\varepsilon_{n\varphi} = s_{44}\,\sigma_{n\varphi} = \frac{\partial C_1}{\partial \varphi} x_n^{\lambda_1 - 1} + \frac{\partial C_2}{\partial \varphi} x_n^{\lambda_2 - 1}, \tag{3.9}$$

$$\varepsilon_{n\theta} = s_{44}\,\sigma_{n\theta} = \Theta \left(\frac{\partial C_1}{\partial \nu} x_n^{\lambda_1 - 1} + \frac{\partial C_2}{\partial \nu} x_n^{\lambda_2 - 1} \right), \tag{3.10}$$

$$\sigma_n = C_1 \xi_1 x_n^{\lambda_1 - 1} + C_2 \xi_2 x_n^{\lambda_2 - 1}, \tag{3.11}$$

$$\sigma_\varphi = \sigma_\theta = C_1 \xi_3 x_n^{\lambda_1 - 1} + C_2 \xi_4 x_n^{\lambda_2 - 1}, \tag{3.12}$$

$$\sigma_1 = \eta_1 x_n^{\lambda_1 - 1} + \eta_2 x_n^{\lambda_2 - 1}, \tag{3.13}$$

$$w = \kappa_1 x_n^{2(\lambda_1 - 1)} + \kappa_2 x_n^{2(\lambda_2 - 1)} + + \kappa_3 x_n^{\lambda_1 + \lambda_2 - 2}, \tag{3.14}$$

where Θ, s_{44} is given by Equations (1.15), (2.13), respectively, and ξ_i, ξ_{2+i}, ξ_{2+i+2j}, η_i, κ_j ($i = 1,2$; $j = 1,2,3$) are derived as

$$\begin{aligned}
&\xi_i = \frac{E\left[\lambda_i (1-\mu) + 2\mu\right]}{(1+\mu)(1-2\mu)}, \quad \xi_{2+i} = \frac{E\,(1+\lambda_i \mu)}{(1+\mu)(1-2\mu)}, \\
&\xi_{2+i+2j} = \frac{E\left\{\lambda_i \left[\lambda_j (1-\mu) + 4\mu\right] + 2\right\}}{2\,(1+\mu)(1-2\mu)}, \\
&\eta_i = C_i\,(\lambda_i \gamma_1 + \gamma_2) + \frac{1}{s_{44}} \left(\gamma_3 \frac{\partial C_i}{\partial \varphi} + \gamma_4 \frac{\partial C_i}{\partial \nu} \right), \\
&\kappa_i = C_i^2 \xi_{2+3i} + \frac{1}{s_{44}} \left[\left(\frac{\partial C_i}{\partial \varphi} \right)^2 + \Theta^2 \left(\frac{\partial C_i}{\partial \nu} \right)^2 \right], \\
&\kappa_3 = C_1 C_2 (\xi_6 + \xi_7) + \frac{1}{s_{44}} \left(\frac{\partial C_1}{\partial \varphi} \frac{\partial C_2}{\partial \varphi} + \Theta^2 \frac{\partial C_1}{\partial \nu} \frac{\partial C_2}{\partial \nu} \right), \quad i, j = 1, 2.
\end{aligned} \tag{3.15}$$

3.2 Matrix

With regard to Equations (2.30), (2.31), (3.6)–(3.14), (2.21), (2.26), (2.27), we get

$$\varepsilon_{nM} = -p_n \left[\frac{\lambda_{1M}}{\zeta_1} \left(\frac{x_n}{x_M} \right)^{\lambda_{1M}-1} + \frac{\lambda_{2M}}{\zeta_2} \left(\frac{x_n}{x_M} \right)^{\lambda_{2M}-1} \right], \tag{3.16}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -p_n \left[\frac{1}{\zeta_1} \left(\frac{x_n}{x_M} \right)^{\lambda_{1M}-1} + \frac{1}{\zeta_2} \left(\frac{x_n}{x_M} \right)^{\lambda_{2M}-1} \right], \tag{3.17}$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = -\frac{\partial}{\partial \varphi} \left(\frac{p_n}{\zeta_1 x_n^{\lambda_{1M}-1}} \right) \left(\frac{x_n}{x_M} \right)^{\lambda_{1M}-1} - \frac{\partial}{\partial \varphi} \left(\frac{p_n}{\zeta_2 x_n^{\lambda_{2M}-1}} \right) \left(\frac{x_n}{x_M} \right)^{\lambda_{2M}-1}, \tag{3.18}$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = -\Theta \left[\frac{\partial}{\partial \nu} \left(\frac{p_n}{\zeta_1 x_n^{\lambda_{1M}-1}} \right) \left(\frac{x_n}{x_M} \right)^{\lambda_{1M}-1} + \frac{\partial}{\partial \nu} \left(\frac{p_n}{\zeta_2 x_n^{\lambda_{2M}-1}} \right) \left(\frac{x_n}{x_M} \right)^{\lambda_{2M}-1} \right], \tag{3.19}$$

$$\sigma_{nM} = -p_n \left[\frac{\xi_{1M}}{\zeta_1} \left(\frac{x_n}{x_M} \right)^{\lambda_{1M}-1} + \frac{\xi_{2M}}{\zeta_2} \left(\frac{x_n}{x_M} \right)^{\lambda_{2M}-1} \right], \tag{3.20}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -p_n \left[\frac{\xi_{3M}}{\zeta_1} \left(\frac{x_n}{x_M} \right)^{\lambda_{1M}-1} + \frac{\xi_{4M}}{\zeta_2} \left(\frac{x_n}{x_M} \right)^{\lambda_{2M}-1} \right], \tag{3.21}$$

$$\sigma_{1M} = \eta_{1M} x_n^{\lambda_{1M}-1} + \eta_{2M} x_n^{\lambda_{2M}-1}, \tag{3.22}$$

$$w_M = \kappa_{1M} x_n^{2(\lambda_{1M}-1)} + \kappa_{2M} x_n^{2(\lambda_{2M}-1)} + +\kappa_{3M} x_n^{\lambda_{1M}+\lambda_{2M}-2}, \tag{3.23}$$

$$W_M = 4 \int_0^{\pi/2} \int_0^{\pi/2} \left[\frac{\kappa_{1M} \left(x_M^{2\lambda_{1M}+1} - x_{IN}^{2\lambda_{1M}+1} \right)}{2\lambda_{1M}+1} + \frac{\kappa_{2M} \left(x_M^{2\lambda_{2M}+1} - x_{IN}^{2\lambda_{2M}+1} \right)}{2\lambda_{2M}+1} + \frac{\kappa_{3M} \left(x_M^{\lambda_{1M}+\lambda_{2M}+1} - x_{IN}^{\lambda_{1M}+\lambda_{2M}+1} \right)}{\lambda_{1M}+\lambda_{2M}+1} \right] d\varphi\, d\nu, \tag{3.24}$$

where Θ, x_{IN}, x_M, s_{44M}, λ_{iM}, ξ_{jM} ($i=1,2;\ j=1,\ldots,8$) are given by Equations (1.15), (1.16), (1.17), (2.13), (3.4), (3.15), respectively, and ζ_i, η_{iM}, κ_{jM} ($i=1,2;\ j=1,2,3;$ see Equation (3.15)) have the forms

$$\zeta_i = \xi_{iM}\left(\frac{x_{IN}}{x_M}\right)^{\lambda_{iM}-1} - \xi_{3-iM}\left(\frac{x_{IN}}{x_M}\right)^{\lambda_{3-iM}-1},$$

$$\eta_{iM} = -\frac{p_n\left(\lambda_{iM}\gamma_{1M}+\gamma_{2M}\right)}{\zeta_i x_n^{\lambda_{iM}-1}} - \frac{\gamma_{3M}}{s_{44M}}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_i x_n^{\lambda_{iM}-1}}\right) - \frac{\gamma_{4M}}{s_{44M}}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_i x_n^{\lambda_{iM}-1}}\right),$$

$$\kappa_{iM} = \xi_{2+3iM}\left(\frac{p_n}{\zeta_i x_M^{\lambda_{iM}-1}}\right)^2 + \frac{1}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_i x_M^{\lambda_{iM}-1}}\right)\right]^2 + \frac{\Theta^2}{s_{44M}}\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_i x_M^{\lambda_{iM}-1}}\right)\right]^2,$$

$$\kappa_{3M} = \frac{p_n^2\left(\xi_{6M}+\xi_{7M}\right)}{\zeta_1\zeta_2 x_M^{\lambda_{1M}+\lambda_{2M}-2}} + \frac{1}{s_{44M}}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_1 x_M^{\lambda_{1M}-1}}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_2 x_M^{\lambda_{2M}-1}}\right) + \frac{\Theta^2}{s_{44M}}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_1 x_M^{\lambda_{1M}-1}}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_2 x_M^{\lambda_{2M}-1}}\right), \quad i=1,2. \tag{3.25}$$

The normal stress p_n is given by Equation (2.33). With regard to Equation (3.17), the coefficient ρ_M in Equation (2.33) is derived as

$$\rho_M = \frac{(1+\mu)(1-2\mu)}{E}\left[\frac{1}{\lambda_1(1-\mu)+2\mu} + \frac{1}{\lambda_2(1-\mu)+2\mu}\right]. \tag{3.26}$$

3.3 Inclusion

In case of the ellipsoidal inclusion, we get $C_{2IN}=0$, otherwise we get $\left(u_{nIN}\right)_{x_n\to 0} \longrightarrow \pm\infty$, $\left(\varepsilon_{IN}\right)_{x_n\longrightarrow 0} \longrightarrow \pm\infty$, $\left(\sigma_{IN}\right)_{r\to 0} \longrightarrow \pm\infty$ due to $\lambda_2 < -2$ (see Equations (3.4), (3.6)–(3.12)). With regard to Equations (2.28), (2.29), (3.6)–(3.14), (2.21), (2.26), (2.27), we get [1]–[22]

$$\varepsilon_{nIN} = -\frac{p_n\lambda_{1IN}}{\xi_{1IN}}\left(\frac{x_n}{x_{IN}}\right)^{\lambda_{1IN}-1}, \tag{3.27}$$

$$\varepsilon_{\varphi IN} = \varepsilon_{\theta IN} = -\frac{p_n}{\xi_{1IN}} \left(\frac{x_n}{x_{IN}} \right)^{\lambda_{1IN}-1}, \tag{3.28}$$

$$\varepsilon_{n\varphi IN} = s_{44IN} \sigma_{r\varphi IN} = -\frac{\partial}{\partial \varphi} \left(\frac{p_n}{\xi_{1IN}\, x_{IN}^{\lambda_{1IN}-1}} \right) x_n^{\lambda_{1IN}-1}, \tag{3.29}$$

$$\varepsilon_{n\nu IN} = s_{44IN} \sigma_{r\nu IN} = -\Theta \frac{\partial}{\partial \nu} \left(\frac{p_n}{\xi_{1IN}\, x_{IN}^{\lambda_{1IN}-1}} \right) x_n^{\lambda_{1IN}-1}, \tag{3.30}$$

$$\sigma_{11IN} = -p_n \left(\frac{x_n}{x_{IN}} \right)^{\lambda_{1IN}-1}, \tag{3.31}$$

$$\sigma_{\varphi IN} = \sigma_{\nu IN} = -\frac{p_n \xi_{3IN}}{\xi_{1IN}} \left(\frac{x_n}{x_{IN}} \right)^{\lambda_{1IN}-1}, \tag{3.32}$$

$$\sigma_{1IN} = \eta_{1IM} x_n^{\lambda_{1IM}-1}, \tag{3.33}$$

$$w_{IN} = \kappa_{1IN} x_n^{2(\lambda_{1IN}-1)} \tag{3.34}$$

$$W_{IN} = \frac{4}{2\lambda_{1IN}+1} \int_0^{\pi/2} \int_0^{\pi/2} \kappa_{1IN}\, x_{IN}^{2\lambda_{1IN}+1}\, d\varphi\, d\nu, \tag{3.35}$$

where Θ, x_{IN}, s_{44IN}, λ_{1IN} and ξ_{1IN}, ξ_{3IN}, ξ_{5IN} are given by Equations (1.15), (1.16), (2.13), (3.4) and η_{1IN} κ_{1IN} (see Equation (3.15)) has the form

$$\eta_{1IN} = -\frac{p_n (\lambda_{1IN}\gamma_{1IN} + \gamma_{2IN})}{\xi_{1IN} x_n^{\lambda_{1IN}-1}} - \frac{\gamma_{3IN}}{s_{44IN}} \frac{\partial}{\partial \varphi} \left(\frac{p_n}{\xi_{1IN} x_n^{\lambda_{1IN}-1}} \right) - \frac{\gamma_{4IN}}{s_{44IN}} \frac{\partial}{\partial \nu} \left(\frac{p_n}{\xi_{1IN} x_n^{\lambda_{1IN}-1}} \right),$$

$$\kappa_{1IN} = \xi_{5IN} \left(\frac{p_n}{\xi_{1IN}\, x_{IN}^{\lambda_{1IN}-1}} \right)^2 + \frac{1}{s_{44IN}} \left[\frac{\partial C_1}{\partial \varphi} \left(\frac{p_n}{\xi_{1IN}\, x_{IN}^{\lambda_{1IN}-1}} \right) \right]^2 + \frac{\Theta^2}{s_{44IN}} \left[\frac{\partial C_1}{\partial \nu} \left(\frac{p_n}{\xi_{1IN}\, x_{IN}^{\lambda_{1IN}-1}} \right) \right]^2. \tag{3.36}$$

The normal stress p_n is given by Equation (2.33). With regard to Equation (3.28), the coefficient ρ_{IN} in Equation (2.33) is derived as

$$\rho_{IN} = \frac{(1+\mu_{IN})(1-2\mu_{IN})}{E_{IN}\left[\lambda_{1IN}(1-\mu_{IN})+2\mu_{IN}\right]}. \tag{3.37}$$

Chapter 4

Mathematical Model 2

4.1 Mathematical Procedure

Let the mathematical procedure $\partial^2 \mathrm{Eq.}(2.24)/\partial x_n^2$ be performed, and then we get [1]–[22]

$$x_n \frac{\partial^3 U_n}{\partial x_n^3} + (2 - c_3)\frac{\partial^2 U_n}{\partial x_n^2} = 0, \tag{4.1}$$

where $c_3 < 0$ and $U_n = U_n(x_n, \varphi, \nu)$ are given by Equations (2.18) and (2.25), respectively. Let U_b be assumed in the form $U_n = x_n^{\lambda}$, and then we get

$$U_n = C_1 x_n + C_2 x_n^{c_3} + C_3, \tag{4.2}$$

where C_1, C_2, C_3 are integration constants, which are determined by the boundary conditions in Section 2.3. Let Equation (2.28) be substituted to Equation (2.23), and then we get

$$x_n^2 \frac{\partial^2 u_n}{\partial x_n^2} + 2x_n \frac{\partial u_n}{\partial x_n} - 2u_n = C_1 x_n + C_2 x_n^{c_3} + C_3 x_n^2. \tag{4.3}$$

The mathematical solution of Equation (4.3), which is determined by Wronskian's method (see Chapter 10) [23], is derived as

$$u_n = C_1 x_n \left(\frac{1}{3} - \ln x_n\right) + C_2 x_n^{c_3} + C_3. \tag{4.4}$$

With regard to Equations (2.1)–(2.4), (2.14)–(2.17), (2.26), (4.4), we get

$$\varepsilon_n = -C_1\left(\frac{2}{3} + \ln x_n\right) + C_2 c_3 x_n^{c_3 - 1}, \tag{4.5}$$

$$\varepsilon_\varphi = \varepsilon_\theta = C_1\left(\frac{1}{3} - \ln x_n\right) + C_2 x_n^{c_3 - 1} + \frac{C_3}{x_n}, \tag{4.6}$$

$$\varepsilon_{n\varphi} = s_{44}\sigma_{n\varphi} = \left(\frac{1}{3} - \ln x_n\right)\frac{\partial C_1}{\partial \varphi} + x_n^{c_3 - 1}\frac{\partial C_2}{\partial \varphi} + \frac{1}{x_n}\frac{\partial C_3}{\partial \varphi}, \tag{4.7}$$

$$\varepsilon_{n\theta} = s_{44}\,\sigma_{n\theta} = \Theta\left[\left(\frac{1}{3} - \ln x_n\right)\frac{\partial C_1}{\partial \nu} + x_n^{c_3-1}\,\frac{\partial C_2}{\partial \nu} + \frac{1}{x_n}\,\frac{\partial C_3}{\partial \nu}\right], \tag{4.8}$$

$$\sigma_n = -\,C_1\left[\frac{2\,(c_1+2\,c_2)}{3} + (c_1-c_2)\ln x_n\right] + C_2\left[(c_1+c_2)\,c_3 - 2\,c_2\right]x_n^{c_3-1} - \frac{2\,C_3\,c_2}{x_n}, \tag{4.9}$$

$$\sigma_\varphi = \sigma_\theta = C_1\,\xi_1\left[\frac{c_1+2\,c_2}{3} - (c_1-c_2)\ln x_n\right] + C_2\,\xi_2\,(c_1 - c_2\,c_3)\,x_n^{c_3-1} + \frac{C_3\,\xi_3\,c_1}{x_n}, \tag{4.10}$$

$$\sigma_1 = \eta_1 + \eta_2\,\ln x_n + \eta_3\,x_n^{c_3-1} + \frac{\eta_4}{x_n}, \tag{4.11}$$

$$\begin{aligned} w &= C_1^2\,\kappa_1 + C_2^2\,\kappa_2 + C_3^2\,\kappa_3 + C_1\,C_2\,\kappa_4 + C_1\,C_3\,\kappa_5 + C_2\,C_3\,\kappa_6 \\ &+ \frac{\chi_1}{s_{44}}\left[\left(\frac{\partial C_1}{\partial \varphi}\right)^2 + \Theta^2\left(\frac{\partial C_1}{\partial \nu}\right)^2\right] + \frac{\chi_2}{s_{44}}\left[\left(\frac{\partial C_2}{\partial \varphi}\right)^2 + \Theta^2\left(\frac{\partial C_2}{\partial \nu}\right)^2\right] \\ &+ \frac{\chi_3}{s_{44}}\left[\left(\frac{\partial C_3}{\partial \varphi}\right)^2 + \Theta^2\left(\frac{\partial C_3}{\partial \nu}\right)^2\right] + \frac{\chi_4}{s_{44}}\left(\frac{\partial C_1}{\partial \varphi}\,\frac{\partial C_2}{\partial \varphi} + \Theta^2\frac{\partial C_1}{\partial \nu}\,\frac{\partial C_2}{\partial \nu}\right) \\ &+ \frac{\chi_5}{s_{44}}\left(\frac{\partial C_1}{\partial \varphi}\,\frac{\partial C_3}{\partial \varphi} + \Theta^2\frac{\partial C_1}{\partial \nu}\,\frac{\partial C_3}{\partial \nu}\right) + \frac{\chi_6}{s_{44}}\left(\frac{\partial C_2}{\partial \varphi}\,\frac{\partial C_3}{\partial \varphi} + \Theta^2\frac{\partial C_2}{\partial \nu}\,\frac{\partial C_3}{\partial \nu}\right), \end{aligned} \tag{4.12}$$

where Θ, c_i ($i=1,2,3$), s_{44} are given by Equations (1.15), (2.18), (2.13), respectively, and η_j, κ_k, χ_k ($j=1,\ldots,4$; $k=1,\ldots,6$) are derived as

$$\begin{aligned} \eta_1 &= \frac{1}{3}\left[C_1\,(\gamma_2 - 2\,\gamma_1) + \frac{1}{s_{44}}\left(\gamma_3\,\frac{\partial C_1}{\partial \varphi} + \gamma_4\,\frac{\partial C_1}{\partial \nu}\right)\right], \\ \eta_2 &= -\left[C_1\,(\gamma_1+\gamma_2) + \frac{1}{s_{44}}\left(\gamma_3\,\frac{\partial C_1}{\partial \varphi} + \gamma_4\,\frac{\partial C_1}{\partial \nu}\right)\right], \\ \eta_3 &= C_2\,(\gamma_1\,c_3 + \gamma_2) + \frac{1}{s_{44}}\left(\gamma_3\,\frac{\partial C_2}{\partial \varphi} + \gamma_4\,\frac{\partial C_2}{\partial \nu}\right), \\ \eta_4 &= C_3\,\gamma_2 + \frac{1}{s_{44}}\left(\gamma_3\,\frac{\partial C_3}{\partial \varphi} + \gamma_4\,\frac{\partial C_3}{\partial \nu}\right), \\ \kappa_1 &= \frac{c_2-c_1}{2}\,\ln^2 x_n + \frac{2\,(c_2-c_1)}{3}\,\ln x_n + \frac{7c_1+2\,c_2}{9}, \\ \kappa_2 &= \left[\frac{c_3^2\,(c_1+c_2)}{2} + c_1\,(1-2\,c_3)\right]x_n^{2(c_3-1)}, \quad \kappa_3 = \frac{c_1}{x_n^2}, \end{aligned}$$

$$\kappa_4 = c_3\,(c_1 - c_2)\,x_n^{c_3-1}\,\ln x_n + 2\left[c_1 \;-\; \frac{c_3\,(2\,c_1 + c_2)}{3}\right] x_n^{c_3-1},$$

$$\kappa_5 = \frac{2\,c_1}{x_n}, \quad \kappa_6 = 0,$$

$$\chi_1 = \ln^2 x_n - \frac{2}{3}\,\ln x_n + \frac{1}{9}, \quad \chi_2 = x_n^{2(c_3-1)}, \quad \chi_3 = \frac{1}{x_n^2},$$

$$\chi_4 = \frac{2}{3}\,x_n^{c_3-1} - 2x_n^{c_3-1}\ln x_n, \quad \chi_5 = \frac{2}{3x_n} - \frac{2\ln x_n}{x_n}, \quad \chi_6 = x_n^{c_3-2}. \tag{4.13}$$

The integrals Φ_i, Ψ_i of the $\kappa_j = \kappa_j\,(x_n)$, $\chi_j = \chi_j\,(x_n)$ $(i=1,\ldots,6)$, respectively, have the forms

$$\Phi_i = \int\limits_{x_{IN}}^{x_M} \kappa_i x_n^2\, dx_n, \quad \Psi_i = \int\limits_{x_{IN}}^{x_M} \chi_i x_n^2\, dx_n, \quad i = 1,\ldots,6, \tag{4.14}$$

where x_{IN}, x_M are given by Equations (1.16), (1.17), respectively. The integrals are determined by the formulae in Chapter 10 (see Equations (10.10)–(10.12)) and consequently, we get

$$\Phi_1 = \frac{c_2 - c_1}{6}\left\{x_M^3\left[\left(\ln x_M - \frac{1}{3}\right)^2 + \frac{1}{9}\right] - x_{IN}^3\left[\left(\ln x_{IN} - \frac{1}{3}\right)^2 + \frac{1}{9}\right]\right\}$$
$$+\ \frac{2\,(c_2 - c_1)}{9}\left[x_M^3\left(\ln x_M - \frac{1}{3}\right) - x_{IN}^3\left(\ln x_{IN} - \frac{1}{3}\right)\right]$$
$$+\ \frac{(7c_1 + 2\,c_2)\left(x_M^3 - x_{IN}^3\right)}{27},$$

$$\Phi_2 = \frac{1}{2\,c_3 + 1}\left[\frac{c_3^2\,(c_1 + c_2)}{2} + c_1\,(1 - 2\,c_3)\right]\left(x_M^{2\,c_3+1} - x_{IN}^{2\,c_3+1}\right),$$

$$\Phi_3 = c_1\,(x_M - x_{IN}),$$

$$\Phi_4 = \frac{c_3\,(c_1 - c_2)}{c_3 + 2}\left[x_M^{c_3+2}\left(\ln x_M - \frac{1}{c_3 + 2}\right) - x_{IN}^{c_3+2}\left(\ln x_{IN} - \frac{1}{c_3 + 2}\right)\right]$$
$$+\ \frac{2}{c_3 + 2}\left[c_1 \;-\; \frac{c_3\,(2\,c_1 + c_2)}{3}\right]\left(x_M^{c_3+2} - x_{IN}^{c_3+2}\right),$$

$$\Phi_5 = c_1\left(x_M^2 - x_{IN}^2\right), \quad \Phi_6 = 0,$$

$$\Psi_1 = \frac{x_M^3}{3}\left[(\ln x_M - 1)\left(\ln x_M - \frac{1}{3}\right) + \frac{2}{9}\right] - \frac{x_{IN}^3}{3}\left[(\ln x_{IN} - 1)\left(\ln x_{IN} - \frac{1}{3}\right) + \frac{2}{9}\right],$$

$$\Psi_2 = \frac{x_M^{2\,c_3+1} - x_{IN}^{2\,c_3+1}}{2\,c_3 + 1}, \quad \Psi_3 = x_M - x_{IN},$$

$$\Psi_4 = \frac{2}{c_3+2}\left\{x_M^{c_3+2}\left[\frac{c_3+5}{3\,(c_3+2)} - \ln x_M\right] - x_{IN}^{c_3+2}\left[\frac{c_3+5}{3\,(c_3+2)} - \ln x_{IN}\right]\right\},$$

$$\Psi_5 = x_M^2\left(\frac{5}{6} - \ln x_M\right) - x_{IN}^2\left(\frac{5}{6} - \ln x_{IN}\right), \quad \Psi_6 = \frac{x_M^{c_3+1} - x_{IN}^{c_3+1}}{c_3+1}. \tag{4.15}$$

In case of the ellipsoidal inclusion, we get $(u_{nIN})_{x_n \to 0} \longrightarrow \pm\infty$, $(\varepsilon_{IN})_{x_n \longrightarrow 0} \longrightarrow \pm\infty$, $(\sigma_{IN})_{r\to 0} \longrightarrow \pm\infty$ due to $(\ln x_n)_{x_n \to 0} \longrightarrow \pm\infty$ and $(x_n^{c_3})_{x_n \to 0} \longrightarrow \pm\infty$ for $c_3 < 0$ (see Equations (2.18), (4.4)–(4.10)). Accordingly, the mathematical solutions (4.4)–(4.10) are suitable for the matrix.

4.2 Matrix

The integration constants C_{1M}, C_{2M}, C_{3M} for the matrix (see Equation (4.4)) are determined by the boundary conditions (2.30), (2.31) or (2.30)–(2.32). The boundary conditions result in the following combinations of C_{1M}, C_{2M}, C_{3M}. Finally, such a combination is considered to exhibit a minimum value of the elastic energy W_C of the cubic cell (see Equation (2.27)).

Conditions $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} = 0$. With regard to Equations (2.30), (2.31), (4.4)–(4.12), (2.21), (2.26), (2.27), we get

$$\varepsilon_{nM} = \frac{p_n}{\zeta}\left[\frac{2}{3} + \ln x_n + c_{3M}\left(\frac{1}{3} - \ln x_M\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right], \tag{4.16}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta}\left[\frac{1}{3} - \ln x_n - \left(\frac{1}{3} - \ln x_M\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right], \tag{4.17}$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = \left(\ln x_n - \frac{1}{3}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right) + x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta_M x_M^{c_{3M}-1}}\left(\frac{1}{3} - \ln x_M\right)\right], \tag{4.18}$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = \Theta\left\{\left(\ln x_n - \frac{1}{3}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right) + x_n^{c_{3M}-1}\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta_M x_M^{c_{3M}-1}}\left(\frac{1}{3} - \ln x_M\right)\right]\right\}, \tag{4.19}$$

$$\sigma_{nM} = \frac{p_n}{\zeta}\left\{\frac{2\,(c_{1M}+2\,c_{2M})}{3} + (c_{1M}-c_{2M})\ln x_n \right.$$
$$\left. + [(c_{1M}+c_{2M})\,c_{3M} - 2\,c_{2M}]\left(\frac{1}{3} - \ln x_M\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right\}, \quad (4.20)$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\,\frac{p_n}{\zeta}\left[\frac{c_{1M}+2\,c_{2M}}{3} - (c_{1M}-c_{2M})\ln x_n \right.$$
$$\left. - (c_{1M}-c_{2M}c_{3M})\left(\frac{1}{3} - \ln x_M\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right], \quad (4.21)$$

$$\sigma_{1M} = \eta_{1M} + \eta_{2M}\ln x_n + \eta_{3M}\, x_n^{c_{3M}-1}, \quad (4.22)$$

$$w_M = \left(\frac{p_n}{\zeta}\right)^2\left\{\kappa_{1M} + \kappa_{2M}\left(\frac{1-3\ln x_M}{3x_M^{c_{3M}-1}}\right)^2 + \frac{\kappa_{4M}\,(3\ln x_M - 1)}{3x_M^{c_{3M}-1}}\right\}$$
$$+ \frac{\chi_{1M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right]^2\right\}$$
$$+ \frac{\chi_{2M}}{s_{44M}}\left(\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n\,(1-3\ln x_M)}{3\,\zeta\, x_M^{c_{3M}-1}}\right]\right\}^2 + \Theta^2\left\{\frac{\partial}{\partial\nu}\left[\frac{p_n\,(1-3\ln x_M)}{3\,\zeta\, x_M^{c_{3M}-1}}\right]\right\}^2\right)$$
$$+ \frac{\chi_{4M}}{s_{44M}}\;\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n\,(3\ln x_M - 1)}{3\,\zeta\, x_M^{c_{3M}-1}}\right]$$
$$+ \frac{\chi_{4M}\,\Theta^2}{s_{44M}}\;\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\nu}\left[\frac{p_n\,(3\ln x_M - 1)}{3\,\zeta\, x_M^{c_{3M}-1}}\right], \quad (4.23)$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{p_n}{\zeta}\right)^2\left\{\Phi_{1M} + \Phi_{2M}\left(\frac{1-3\ln x_M}{3x_M^{c_{3M}-1}}\right)^2 + \frac{\Phi_{4M}\,(3\ln x_M - 1)}{3x_M^{c_{3M}-1}}\right\} d\varphi\, d\nu$$
$$+ \frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{1M}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right]^2\right\} d\varphi\, d\nu$$
$$+ \frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{2M}\left(\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n\,(1-3\ln x_M)}{3\,\zeta\, x_M^{c_{3M}-1}}\right]\right\}^2\right) d\varphi\, d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{2M}\,\Theta^2\left(\left\{\frac{\partial}{\partial\nu}\left[\frac{p_n\,(1-3\ln x_M)}{3\,\zeta\,x_M^{c_{3M}-1}}\right]\right\}^2\right)d\varphi\,d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{4M}\;\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n\,(3\ln x_M-1)}{3\,\zeta\,x_M^{c_{3M}-1}}\right]d\varphi\,d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{4M}\,\Theta^2\;\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\nu}\left[\frac{p_n\,(3\ln x_M-1)}{3\,\zeta\,x_M^{c_{3M}-1}}\right]d\varphi\,d\nu, \tag{4.24}$$

where Θ, x_M, s_{44M}, c_{iM} $(i=1{,}2{,}3)$ and κ_{jM}, χ_{jM}; Φ_{jM}, Ψ_{jM} $(j=1{,}2{,}4)$ are given by Equations (1.15), (1.17), (2.13), (2.18) and (4.13); (4.15), respectively, and ζ, ζ_i $(i=1{,}2)$, η_{jM} ($j=1{,}2{,}3$; see Equation (4.13)) have the forms

$$\zeta=\zeta_2-\zeta_1\left(\frac{1}{3}-\ln x_M\right),\quad \zeta_1=\left[(c_{1M}+c_{2M})\,c_{3M}-2\,c_{2M}\right]\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1},$$

$$\zeta_2=-\left[\frac{2\,(c_{1M}+2\,c_{2M})}{3}+(c_{1M}-c_{2M})\ln x_{IN}\right],$$

$$\eta_{1M}=-\frac{1}{3}\left\{\frac{p_n\,(\gamma_{2m}-2\,\gamma_{1M})}{\zeta}+\frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)+\gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right]\right\},$$

$$\eta_{2M}=\left\{\frac{p_n\,(\gamma_{1M}+\gamma_{2M})}{\zeta}+\frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)+\gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right]\right\},$$

$$\eta_{3M}=\frac{p_n\,(\gamma_{1M}\,c_{3M}+\gamma_{2M})}{\zeta_M x_M^{c_{3M}-1}}\left(\frac{1}{3}-\ln x_M\right)+\frac{\gamma_{3M}}{s_{44M}}\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta_M x_M^{c_{3M}-1}}\left(\frac{1}{3}-\ln x_M\right)\right]$$
$$+\frac{\gamma_{4M}}{s_{44M}}\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta_M x_M^{c_{3M}-1}}\left(\frac{1}{3}-\ln x_M\right)\right]. \tag{4.25}$$

The normal stress p_n is given by Equation (2.33). With regard to Equation (4.20), the coefficient ρ_M in Equation (2.33) is derived as

$$\rho_M=\frac{1}{\zeta}\left[\frac{1}{3}-\ln x_{IN}-\left(\frac{1}{3}-\ln x_M\right)\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1}\right]. \tag{4.26}$$

Conditions $C_{1M}\neq 0$, $C_{3M}\neq 0$, $C_{2M}=0$. With regard to Equations (2.30), (2.31), (4.4)–(4.12), (2.21), (2.26), (2.27), we get

$$\varepsilon_{nM}=\frac{p_n}{\zeta\,x_M}\left(\frac{2}{3}+\ln x_n\right), \tag{4.27}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta}\left[\frac{1}{x_M}\left(\frac{1}{3} - \ln x_n\right) - \frac{1}{x_n}\left(\frac{1}{3} - \ln x_M\right)\right], \tag{4.28}$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = -\left(\frac{1}{3} - \ln x_n\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M}\right) + \frac{1}{x_n}\frac{\partial}{\partial\varphi}\left[\frac{p_n\,(1 - 3\ln x_M)}{3\,\zeta}\right], \tag{4.29}$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = -\Theta\left\{\left(\frac{1}{3} - \ln x_n\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M}\right) - \frac{1}{x_n}\frac{\partial}{\partial\varphi}\left[\frac{p_n\,(1 - 3\ln x_M)}{3\,\zeta}\right]\right\}, \tag{4.30}$$

$$\sigma_{nM} = \frac{p_n}{\zeta}\left\{\frac{1}{x_M}\left[\frac{2\,(c_{1M} + 2\,c_{2M})}{3} + (c_{1M} - c_{2M})\ln x_n\right] - \frac{2\,c_{2M}}{x_n}\left(\frac{1}{3} - \ln x_M\right)\right\}, \tag{4.31}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta}\left\{\frac{1}{x_M}\left[\frac{c_{1M} + 2\,c_{2M}}{3} - (c_{1M} - c_{2M})\ln x_n\right] - \frac{c_{1M}}{x_n}\left(\frac{1}{3} - \ln x_M\right)\right\}, \tag{4.32}$$

$$\sigma_{1M} = \eta_{1M} + \eta_{2M}\ln x_n + \frac{\eta_{4M}}{x_n}, \tag{4.33}$$

$$\begin{aligned} w_M = {} & \left(\frac{p_n}{\zeta}\right)^2\left[\frac{\kappa_{1M}}{x_M^2} + \kappa_{3M}\left(\frac{1}{3} - \ln x_M\right)^2 + \frac{\kappa_{5M}\,(3\ln x_M - 1)}{3x_M}\right] \\ & + \frac{\chi_{1M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M}\right)\right]^2\right\} \\ & + \frac{\chi_{3M}}{s_{44M}}\left(\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n\,(1 - 3\ln x_M)}{3\,\zeta}\right]\right\}^2 + \Theta^2\left\{\frac{\partial}{\partial\nu}\left[\frac{p_n\,(1 - 3\ln x_M)}{3\,\zeta}\right]\right\}^2\right) \\ & + \frac{\chi_{5M}}{s_{44M}}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M}\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n\,(3\ln x_M - 1)}{3\,\zeta}\right] \\ & + \frac{\chi_{5M}\,\Theta^2}{s_{44M}}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M}\right)\frac{\partial}{\partial\nu}\left[\frac{p_n\,(3\ln x_M - 1)}{3\,\zeta}\right], \end{aligned} \tag{4.34}$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{p_n}{\zeta}\right)^2\left[\frac{\Phi_{1M}}{x_M^2} + \Phi_{3M}\left(\frac{1}{3} - \ln x_M\right)^2 + \frac{\Phi_{5M}\,(3\ln x_M - 1)}{3x_M}\right]d\varphi\, d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{1M}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M}\right)\right]^2\right\}d\varphi\, d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{3M}\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n\,(1-3\ln x_M)}{3\,\zeta}\right]\right\}^2 d\varphi\, d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{3M}\,\Theta^2\left\{\frac{\partial}{\partial\nu}\left[\frac{p_n\,(1-3\ln x_M)}{3\,\zeta}\right]\right\}^2 d\varphi\, d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{5M}\,\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M}\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n\,(3\ln x_M-1)}{3\,\zeta}\right]d\varphi\, d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{5M}\,\Theta^2\,\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M}\right)\frac{\partial}{\partial\nu}\left[\frac{p_n\,(3\ln x_M-1)}{3\,\zeta}\right]d\varphi\, d\nu, \quad (4.35)$$

where Θ, x_M, s_{44M}, c_{iM} ($i=1,2,3$) and κ_{jM}, χ_{jM}, Φ_{jM}, Ψ_{jM} ($j=1,3,5$) are given by Equations (1.15), (1.17), (2.13), (2.18) and (4.13); (4.15), respectively, and ζ, η_{iM} ($i=1,2,4$; see Equation (4.13)) have the forms

$$\zeta=\frac{1}{x_M}\left[\frac{2\,(c_{1M}+2\,c_{2M})}{3}+(c_{1M}-c_{2M})\ln x_{IN}\right]+\frac{2\,c_{2M}}{x_{IN}}\left(\frac{1}{3}-\ln x_M\right),$$

$$\eta_{1M}=-\frac{1}{3}\left\{\frac{p_n\,(\gamma_{2M}-2\,\gamma_{1M})}{\zeta\, x_M}+\frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M}\right)+\gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M}\right)\right]\right\},$$

$$\eta_{2M}=\frac{p_n\,(\gamma_{1M}+\gamma_{2M})}{\zeta\, x_M}+\frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M}\right)+\gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M}\right)\right],$$

$$\eta_{4M}=\frac{p_n\,\gamma_{2M}\,(1-3\ln x_M)}{3\,\zeta}+\frac{\gamma_{3M}}{s_{44M}}\frac{\partial}{\partial\varphi}\left[\frac{p_n\,(1-3\ln x_M)}{3\,\zeta}\right]$$
$$+\frac{\gamma_{4M}}{s_{44M}}\frac{\partial}{\partial\nu}\left[\frac{p_n\,(1-3\ln x_M)}{3\,\zeta}\right]. \quad (4.36)$$

The normal stress p_n is given by Equation (2.33). With regard to Equation (4.28), the coefficient ρ_M in Equation (2.33) is derived as

$$\rho_M=\frac{1}{\zeta}\left[\frac{1}{x_M}\left(\frac{1}{3}-\ln x_{IN}\right)-\frac{1}{x_{IN}}\left(\frac{1}{3}-\ln x_M\right)\right]. \quad (4.37)$$

Conditions $C_{2M} \neq 0$**,** $C_{3M} \neq 0$**,** $C_{1M} = 0$**.** With regard to Equations (2.30), (2.31), (4.4)–(4.12), (2.21), (2.26), (2.27), we get

$$\varepsilon_{nM} = -\frac{p_n c_{3M}}{\zeta x_M}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}, \tag{4.38}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = \frac{p_n}{\zeta x_n}\left[1 - \left(\frac{x_n}{x_M}\right)^{c_{3M}}\right], \tag{4.39}$$

$$\varepsilon_{n\varphi M} = s_{44M}\sigma_{n\varphi M} = -\frac{1}{x_n}\left[x_n^{c_{3M}}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta x_M^{c_{3M}}}\right) - \frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right], \tag{4.40}$$

$$\varepsilon_{n\theta M} = s_{44M}\sigma_{n\theta M} = -\frac{\Theta}{x_n}\left[x_n^{c_{3M}}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta x_M^{c_{3M}}}\right) - \frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right], \tag{4.41}$$

$$\sigma_{nM} = -\frac{p_n}{\zeta}\left\{\frac{c_{3M}(c_{1M}+c_{2M}) - 2c_{2M}}{x_M}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} + \frac{2c_{2M}}{x_n}\right\}, \tag{4.42}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta}\left[\frac{c_{1M} - c_{2M}c_{3M}}{x_M}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} - \frac{c_{1M}}{x_n}\right], \tag{4.43}$$

$$\sigma_{1M} = \eta_{3M}x_n^{c_{3M}-1} + \frac{\eta_{4M}}{x_n}, \tag{4.44}$$

$$\begin{aligned} w_M = {} & \left(\frac{p_n}{\zeta}\right)^2\left(\frac{\kappa_{2M}}{x_M^{2c_{3M}}} + \kappa_{3M} - \frac{\kappa_{6M}}{x_M^{c_{3M}}}\right) \\ & + \frac{\chi_{2M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta x_M^{c_{3M}}}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta x_M^{c_{3M}}}\right)\right]^2\right\} \\ & + \frac{\chi_{3M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right]^2\right\} \\ & - \frac{\chi_{6M}}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta x_M^{c_{3M}}}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right) + \Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta x_M^{c_{3M}}}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right], \end{aligned} \tag{4.45}$$

$$
\begin{aligned}
W_M = {} & 4\int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{p_n}{\zeta}\right)^2\left(\frac{\Phi_{2M}}{x_M^{2c_{3M}}}+\Phi_{3M}-\frac{\Phi_{6M}}{x_M^{c_{3M}}}\right)d\varphi\,d\nu \\
& +\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{2M}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\,x_M^{c_{3M}}}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\,x_M^{c_{3M}}}\right)\right]^2\right\}d\varphi\,d\nu \\
& +\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{3M}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right]^2\right\}d\varphi\,d\nu \\
& -\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{6M}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\,x_M^{c_{3M}}}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)+\Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\,x_M^{c_{3M}}}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right]d\varphi\,d\nu,
\end{aligned}
\tag{4.46}
$$

where Θ, x_M, s_{44M}, c_{iM} $(i=1,2,3)$ and κ_{jM}, χ_{jM}; Φ_{jM}, Ψ_{jM} $(j=2,3,6)$ are given by Equations (1.15), (1.17), (2.13), (2.18) and (4.13); (4.15), respectively, and ζ, η_{iM} $(i=2,3$; see Equation (4.13)) have the forms

$$
\begin{aligned}
\zeta &= \frac{1}{x_{IN}}\left\{\left[c_{3M}\left(c_{1M}+c_{2M}\right)-2c_{2M}\right]\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}}+2c_{2M}\right\}, \\
\eta_{3M} &= -\frac{p_n\left(\gamma_{1M}c_{3M}+\gamma_{2M}\right)}{\zeta\,x_M^{c_{3M}}}-\frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\,x_M^{c_{3M}}}\right)-\gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\,x_M^{c_{3M}}}\right)\right], \\
\eta_{4M} &= \frac{p_n\gamma_2}{\zeta}+\frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)+\gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right].
\end{aligned}
\tag{4.47}
$$

The normal stress p_n is given by Equation (2.33). With regard to Equation (4.39), the coefficient ρ_M in Equation (2.33) is derived as

$$
\rho_M=\frac{1}{\zeta}\left[\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}}-1\right]. \tag{4.48}
$$

Conditions $C_{1M}\neq 0$, $C_{2M}\neq 0$, $C_{3M}\neq 0$. With regard to Equations (2.30)–(2.31), (4.4)–(4.12), (2.21), (2.26), (2.27), we get

$$
\varepsilon_{nM}=-\frac{p_n}{\zeta}\left[\zeta_1\left(\frac{2}{3}+\ln x_n\right)-\zeta_2\,c_{3M}x_n^{c_{3M}-1}\right], \tag{4.49}
$$

$$
\varepsilon_{\varphi M}=\varepsilon_{\theta M}=\frac{p_n}{\zeta}\left[\zeta_1\left(\frac{1}{3}-\ln x_n\right)+\zeta_2 x_n^{c_{3M}-1}+\frac{\zeta_3}{x_n}\right], \tag{4.50}
$$

$$\varepsilon_{n\varphi M} = s_{44M}\sigma_{n\varphi M} = \left(\frac{1}{3}-\ln x_n\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right) + x_n^{c_3-1}\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right) + \frac{1}{x_n}\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right), \tag{4.51}$$

$$\varepsilon_{n\theta M} = s_{44M}\sigma_{n\theta M} = \Theta\left[\left(\frac{1}{3}-\ln x_n\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right) + x_n^{c_3-1}\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right) + \frac{1}{x_n}\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right], \tag{4.52}$$

$$\sigma_{nM} = -\frac{p_n}{\zeta}\left\{\zeta_1\left[\frac{2(c_{1M}+2c_{2M})}{3} + (c_{1M}-c_{2M})\ln x_n\right] - \zeta_2\left[(c_{1M}+c_{2M})c_{3M}-2c_{2M}\right]x_n^{c_{3M}-1} + \frac{2c_{2M}\zeta_3}{x_n}\right\}, \tag{4.53}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = \frac{p_n}{\zeta}\left\{\zeta_1\left[\frac{c_{1M}+2c_{2M}}{3} - (c_{1M}-c_{2M})\ln x_n\right] + \zeta_2(c_{1M}-c_{2M}c_{3M})x_n^{c_{3M}-1} + \frac{c_{1M}\zeta_3}{x_n}\right\}, \tag{4.54}$$

$$\sigma_{1M} = \eta_{1M} + \eta_{2M}\ln x_n + \eta_{3M}x_n^{c_{3M}-1} + \frac{\eta_{4M}}{x_n}, \tag{4.55}$$

$$\begin{aligned} w_M = {} & \left(\frac{p_n}{\zeta}\right)^2\left(\kappa_{1M}\zeta_1^2 + \kappa_{2M}\zeta_2^2 + \kappa_{3M}\zeta_3^2 + \kappa_{4M}\zeta_1\zeta_2 + \kappa_{5M}\zeta_1\zeta_3 + \kappa_{6M}\zeta_2\zeta_3\right) \\ & + \frac{\chi_{1M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\right]^2\right\} \\ & + \frac{\chi_{2M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right)\right]^2\right\} \\ & + \frac{\chi_{3M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]^2\right\} \\ & + \frac{\chi_{4M}}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right) + \Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right)\right] \\ & + \frac{\chi_{5M}}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right) + \Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right] \\ & + \frac{\chi_{6M}}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right) + \Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right], \end{aligned} \tag{4.56}$$

$$
\begin{aligned}
W_M = 4 \int\limits_0^{\pi/2} \int\limits_0^{\pi/2} \left(\frac{p_n}{\zeta} \right)^2 \Big(& \Phi_{1M} \zeta_1^2 + \Phi_{2M} \zeta_2^2 + \Phi_{3M} \zeta_3^2 + \Phi_{4M} \zeta_1 \zeta_2 \\
& + \Phi_{5M} \zeta_1 \zeta_3 + \Phi_{6M} \zeta_2 \zeta_3 \Big) \, d\varphi \, d\nu \\
& + \frac{4}{s_{44M}} \int\limits_0^{\pi/2} \int\limits_0^{\pi/2} \Psi_{1M} \left\{ \left[\frac{\partial}{\partial \varphi} \left(\frac{p_n \zeta_1}{\zeta} \right) \right]^2 + \Theta^2 \left[\frac{\partial}{\partial \nu} \left(\frac{p_n \zeta_1}{\zeta} \right) \right]^2 \right\} d\varphi \, d\nu \\
& + \frac{4}{s_{44M}} \int\limits_0^{\pi/2} \int\limits_0^{\pi/2} \Psi_{2M} \left\{ \left[\frac{\partial}{\partial \varphi} \left(\frac{p_n \zeta_2}{\zeta} \right) \right]^2 + \Theta^2 \left[\frac{\partial}{\partial \nu} \left(\frac{p_n \zeta_2}{\zeta} \right) \right]^2 \right\} d\varphi \, d\nu \\
& + \frac{4}{s_{44M}} \int\limits_0^{\pi/2} \int\limits_0^{\pi/2} \Psi_{3M} \left\{ \left[\frac{\partial}{\partial \varphi} \left(\frac{p_n \zeta_3}{\zeta} \right) \right]^2 + \Theta^2 \left[\frac{\partial}{\partial \nu} \left(\frac{p_n \zeta_3}{\zeta} \right) \right]^2 \right\} d\varphi \, d\nu \\
& + \frac{4}{s_{44M}} \int\limits_0^{\pi/2} \int\limits_0^{\pi/2} \Psi_{4M} \left[\frac{\partial}{\partial \varphi} \left(\frac{p_n \zeta_1}{\zeta} \right) \frac{\partial}{\partial \varphi} \left(\frac{p_n \zeta_2}{\zeta} \right) \right. \\
& \qquad \left. + \Theta^2 \frac{\partial}{\partial \nu} \left(\frac{p_n \zeta_1}{\zeta} \right) \frac{\partial}{\partial \nu} \left(\frac{p_n \zeta_2}{\zeta} \right) \right] d\varphi \, d\nu \\
& + \frac{4}{s_{44M}} \int\limits_0^{\pi/2} \int\limits_0^{\pi/2} \Psi_{5M} \left[\frac{\partial}{\partial \varphi} \left(\frac{p_n \zeta_1}{\zeta} \right) \frac{\partial}{\partial \varphi} \left(\frac{p_n \zeta_3}{\zeta} \right) \right. \\
& \qquad \left. + \Theta^2 \frac{\partial}{\partial \nu} \left(\frac{p_n \zeta_1}{\zeta} \right) \frac{\partial}{\partial \nu} \left(\frac{p_n \zeta_3}{\zeta} \right) \right] d\varphi \, d\nu \\
& + \frac{4}{s_{44M}} \int\limits_0^{\pi/2} \int\limits_0^{\pi/2} \Psi_{6M} \left[\frac{\partial}{\partial \varphi} \left(\frac{p_n \zeta_2}{\zeta} \right) \frac{\partial}{\partial \varphi} \left(\frac{p_n \zeta_3}{\zeta} \right) \right. \\
& \qquad \left. + \Theta^2 \frac{\partial}{\partial \nu} \left(\frac{p_n \zeta_2}{\zeta} \right) \frac{\partial}{\partial \nu} \left(\frac{p_n \zeta_3}{\zeta} \right) \right] d\varphi \, d\nu, \qquad (4.57)
\end{aligned}
$$

where Θ, x_M, s_{44M}, c_{iM} ($i=1,2,3$) and κ_{iM}, χ_{iM}; Φ_{iM}, Ψ_{1M} ($i=1,\ldots,6$) are given by Equations (1.15), ((1.17), (2.13), (2.18) and (4.13); (4.15), respectively, and ζ_i, ζ, η_{iM} ($i=1,2,3$; see Equation (4.13)) have the forms

$$\zeta_1 = c_{3M} x_M^{c_{3M}-1}, \quad \zeta_2 = \frac{2}{3} + \ln x_M, \quad \zeta_3 = -x_M^{c_{3M}} \left[\frac{2}{3} + \ln x_M + c_{3M} \left(\frac{1}{3} - \ln x_M \right) \right],$$

$$\zeta = c_{3M} \left\{ \left[\frac{2\,(c_{1M} + 2\,c_{2M})}{3} + (c_{1M} - c_{2M}) \ln x_{IN} \right] - \frac{2\,c_{2M} x_M}{x_{IN}} \left(\frac{1}{3} - \ln x_M \right) \right\} x_M^{c_{3M}-1}$$
$$- \left\{ [(c_{1M} + c_{2M})\, c_{3M} - 2\,c_{2M}]\, x_{IN}^{c_{3M}-1} + \frac{2\,c_{2M} x_M^{c_{3M}}}{x_{IN}} \right\} \left(\frac{2}{3} + \ln x_M \right),$$

$$\eta_{1M} = \frac{1}{3} \left\{ \frac{p_n \zeta_1 (\gamma_{2M} - 2\gamma_{1M})}{\zeta} + \frac{1}{s_{44m}} \left[\gamma_{3M} \frac{\partial}{\partial \varphi} \left(\frac{p_n \zeta_1}{\zeta} \right) + \gamma_{4M} \frac{\partial}{\partial \nu} \left(\frac{p_n \zeta_1}{\zeta} \right) \right] \right\},$$

$$\eta_{2M} = - \left\{ \frac{p_n \zeta_1 (\gamma_{1M} + \gamma_{2M})}{\zeta} + \frac{1}{s_{44m}} \left[\gamma_{3M} \frac{\partial}{\partial \varphi} \left(\frac{p_n \zeta_1}{\zeta} \right) + \gamma_{4M} \frac{\partial}{\partial \nu} \left(\frac{p_n \zeta_1}{\zeta} \right) \right] \right\},$$

$$\eta_{3M} = \frac{p_n \zeta_2 (\gamma_{1M} c_3 + \gamma_{2M})}{\zeta} + \frac{1}{s_{44m}} \left[\gamma_{3M} \frac{\partial}{\partial \varphi} \left(\frac{p_n \zeta_2}{\zeta} \right) + \gamma_{4M} \frac{\partial}{\partial \nu} \left(\frac{p_n \zeta_2}{\zeta} \right) \right],$$

$$\eta_{4M} = \frac{p_n \zeta_3 \gamma_{2M}}{\zeta} + \frac{1}{s_{44m}} \left[\gamma_{3M} \frac{\partial}{\partial \varphi} \left(\frac{p_n \zeta_3}{\zeta} \right) + \gamma_{4M} \frac{\partial}{\partial \nu} \left(\frac{p_n \zeta_3}{\zeta} \right) \right]. \qquad (4.58)$$

The normal stress p_n is given by Equation (2.33). With regard to Equation (4.50), the coefficient ρ_M in Equation (2.33) is derived as

$$\rho_M = \frac{1}{\zeta} \left[\zeta_1 \left(\ln x_{IN} - \frac{1}{3} \right) - \zeta_2 x_{IN}^{c_{3M}-1} - \frac{\zeta_3}{x_{IN}} \right]. \qquad (4.59)$$

Chapter 5

Mathematical Model 3

5.1 Mathematical Procedure

Let the mathematical procedure $\partial^2 \mathrm{Eq.}(2.23)/\partial x_n^2$ be performed, and then we get [1]–[22]

$$r\,\frac{\partial^3 u_n}{\partial x_n^3} + 4x_n^2\,\frac{\partial^2 u_n}{\partial x_n^2} + \frac{x_n}{s_{44}\,(c_1 + c_2)}\,\frac{\partial U_n}{\partial x_n} = 0, \tag{5.1}$$

where s_{44}, c_i $(i = 1,2,3)$ and $U_n = U_n\,(r, \varphi, \nu)$ are given by Equations (2.13), (2.18) and (2.25), respectively. With regard to Equations (2.24), (4.2), we get

$$x_n\,\frac{\partial U_n}{\partial x_n} = c_3\,(C_1\,x_n + C_2\,x_n^{c_3} + C_3)\,, \tag{5.2}$$

where C_1, C_2, C_3 are integration constants, which are determined by the boundary conditions in Section 2.3. Let Equation (5.2) be substituted to Equation (5.1), and then we get

$$x_n^3\,\frac{\partial^3 u_n}{\partial x_n^3} + 4x_n^2\,\frac{\partial^2 u_n}{\partial x_n^2} = C_1\,x_n^3 + C_2\,x_n^{c_3} + C_3. \tag{5.3}$$

The mathematical solution of Equation (5.3), which is determined by Wronskian's method (see Chapter 10) [23], is derived as

$$u_n = C_1\,x_n\left(\frac{4}{3} - \ln x_n\right) + C_2\,x_n^{c_3} + C_3\left(\frac{1}{2} + \ln x_n\right). \tag{5.4}$$

With regard to Equations (2.1)–(2.4), (2.14)–(2.17), (2.26), (5.4), we get

$$\varepsilon_n = C_1\left(\frac{1}{3} - \ln x_n\right) + C_2\,c_3\,x_n^{c_3-1} + \frac{C_3}{x_n}, \tag{5.5}$$

$$\varepsilon_\varphi = \varepsilon_\theta = C_1\left(\frac{4}{3} - \ln x_n\right) + C_2\,x_n^{c_3-1} + \frac{C_3}{x_n}\left(\frac{1}{2} + \ln x_n\right), \tag{5.6}$$

$$\varepsilon_{n\varphi} = s_{44}\,\sigma_{n\varphi} = \left(\frac{4}{3} - \ln x_n\right)\frac{\partial C_1}{\partial \varphi} + x_n^{c_3-1}\,\frac{\partial C_2}{\partial \varphi} + \frac{1}{x_n}\left(\frac{1}{2} + \ln x_n\right)\frac{\partial C_3}{\partial \varphi}, \tag{5.7}$$

$$\varepsilon_{n\theta} = s_{44}\,\sigma_{n\theta} = \Theta\left[\left(\frac{4}{3} - \ln x_n\right)\frac{\partial C_1}{\partial \nu} + x_n^{c_3-1}\frac{\partial C_2}{\partial \nu} + \frac{1}{x_n}\left(\frac{1}{2} + \ln x_n\right)\frac{\partial C_3}{\partial \nu}\right], \quad (5.8)$$

$$\sigma_n = C_1\left[\frac{c_1 - 7c_2}{3} - (c_1 - c_2)\ln x_n\right] + C_2\left[(c_1 + c_2)\,c_3 - 2\,c_2\right]x_n^{c_3-1} + \frac{C_3}{x_n}\left(c_1 - 2\,c_2\ln x_n\right), \quad (5.9)$$

$$\sigma_\varphi = \sigma_\theta = C_1\left[\frac{4c_1 - c_2}{3} - (c_1 - c_2)\ln x_n\right] + C_2\,(c_1 - c_2\,c_3)\,x_n^{c_3-1} + \frac{C_3}{x_n}\left(\frac{c_1 - 2\,c_2}{2} + c_1\ln x_n\right), \quad (5.10)$$

$$\sigma_1 = \eta_1 + \eta_2\ln x_n + \eta_3\,x_n^{c_3-1} + \frac{\eta_4 + \eta_5\ln x_n}{x_n}, \quad (5.11)$$

$$\begin{aligned} w = {} & C_1^2\,\kappa_1 + C_2^2\,\kappa_2 + C_3^2\,\kappa_3 + C_1\,C_2\,\kappa_4 + C_1\,C_3\,\kappa_5 + C_2\,C_3\,\kappa_6 \\ & + \frac{\chi_1}{s_{44}}\left[\left(\frac{\partial C_1}{\partial\varphi}\right)^2 + \Theta^2\left(\frac{\partial C_1}{\partial\nu}\right)^2\right] + \frac{\chi_2}{s_{44}}\left[\left(\frac{\partial C_2}{\partial\varphi}\right)^2 + \Theta^2\left(\frac{\partial C_2}{\partial\nu}\right)^2\right] \\ & + \frac{\chi_3}{s_{44}}\left[\left(\frac{\partial C_3}{\partial\varphi}\right)^2 + \Theta^2\left(\frac{\partial C_3}{\partial\nu}\right)^2\right] + \frac{\chi_4}{s_{44}}\left(\frac{\partial C_1}{\partial\varphi}\frac{\partial C_2}{\partial\varphi} + \Theta^2\frac{\partial C_1}{\partial\nu}\frac{\partial C_2}{\partial\nu}\right) \\ & + \frac{\chi_5}{s_{44}}\left(\frac{\partial C_1}{\partial\varphi}\frac{\partial C_3}{\partial\varphi} + \Theta^2\frac{\partial C_1}{\partial\nu}\frac{\partial C_3}{\partial\nu}\right) + \frac{\chi_6}{s_{44}}\left(\frac{\partial C_2}{\partial\varphi}\frac{\partial C_3}{\partial\varphi} + \Theta^2\frac{\partial C_2}{\partial\nu}\frac{\partial C_3}{\partial\nu}\right), \end{aligned} \quad (5.12)$$

where Θ is given by Equation (1.15), and η_i κ_j, χ_j ($i = 1, \ldots, 4;\ j = 1, \ldots, 6$) are derived as

$$\begin{aligned} \eta_1 &= \frac{1}{3}\left[C_1\,(\gamma_1 + 4\,\gamma_2) + \frac{4}{s_{44}}\left(\gamma_3\frac{\partial C_1}{\partial\varphi} + \gamma_4\frac{\partial C_1}{\partial\nu}\right)\right], \\ \eta_2 &= -\left[C_1\,(\gamma_1 + \gamma_2) + \frac{1}{s_{44}}\left(\gamma_3\frac{\partial C_1}{\partial\varphi} + \gamma_4\frac{\partial C_1}{\partial\nu}\right)\right], \\ \eta_3 &= C_2\,(\gamma_1\,c_3 + \gamma_2) + \frac{1}{s_{44}}\left(\gamma_3\frac{\partial C_2}{\partial\varphi} + \gamma_4\frac{\partial C_2}{\partial\nu}\right), \\ \eta_4 &= C_3\left(\gamma_1 + \frac{\gamma_2}{2}\right) + \frac{1}{2\,s_{44}}\left(\gamma_3\frac{\partial C_3}{\partial\varphi} + \gamma_4\frac{\partial C_3}{\partial\nu}\right), \\ \eta_5 &= C_3\,\gamma_2 + \frac{1}{s_{44}}\left(\gamma_3\frac{\partial C_3}{\partial\varphi} + \gamma_4\frac{\partial C_3}{\partial\nu}\right), \end{aligned}$$

$$
\begin{aligned}
&\kappa_1 = \frac{c_2 - c_1}{2} \ln^2 x_n + \frac{c_1 - c_2}{3} \ln x_n + \frac{17\, c_1 + c_2}{18},\\
&\kappa_2 = \left[\frac{c_3^2\,(c_1 + c_2)}{2} + c_1\,(1 - 2\,c_3)\right] x_n^{2(c_3-1)},\\
&\kappa_3 = \frac{c_1 \ln^2 x_n}{x_n^2} - \frac{c_1 \ln x_n}{x_n^2} + \frac{c_2 - 2\,c_1}{4x_n^2},\\
&\kappa_4 = c_3\,(c_1 - c_2)\, x_n^{c_3-1} \ln x_n + \left[2\,c_1 + \frac{c_3\,(c_2 - 7c_1)}{3}\right] x_n^{c_3-1},\\
&\kappa_5 = (3\,c_1 - c_2)\, \frac{\ln x_n}{x_n} - \frac{4\,c_1 - c_2}{3\,x_n},\\
&\kappa_6 = 2\,c_1\,(1 - c_3)\, x_n^{c_3-2} \ln x_n + (c_2\, c_3 - c_1)\, x_n^{c_3-2},\\
&\chi_1 = \ln^2 x_n - \frac{8}{3} \ln x_n + \frac{16}{9}, \quad \chi_2 = x_n^{2(c_3-1)},\\
&\chi_3 = \frac{\ln^2 x_n}{x_n^2} + \frac{\ln x_n}{x_n^2} + \frac{1}{4\,x_n^2}, \quad \chi_4 = \frac{8}{3} x_n^{c_3-1} - 2x_n^{c_3-1} \ln x_n,\\
&\chi_5 = \frac{4}{3x_n} + \frac{5 \ln x_n}{3x_n} - \frac{2 \ln^2 x_n}{x_n}, \quad \chi_6 = 2\,x_n^{c_3-2} \ln x_n + x_n^{c_3-2}.
\end{aligned}
\tag{5.13}
$$

With regard to Equations (4.14), (5.14), we get

$$
\begin{aligned}
\Phi_1 &= \frac{c_2 - c_1}{6} \left\{ x_M^3 \left[\left(\ln x_M - \frac{1}{3}\right)^2 + \frac{1}{9}\right] - x_{IN}^3 \left[\left(\ln x_{IN} - \frac{1}{3}\right)^2 + \frac{1}{9}\right] \right\}\\
&+ \frac{c_1 - c_2}{9} \left[x_M^3 \left(\ln x_M - \frac{1}{3}\right) - x_{IN}^3 \left(\ln x_{IN} - \frac{1}{3}\right)\right] + \frac{17\, c_1 + c_2}{54} \left(x_M^3 - x_{IN}^3\right),\\
\Phi_2 &= \frac{1}{2\,c_3 + 1} \left[\frac{c_3^2\,(c_1 + c_2)}{2} + c_1\,(1 - 2\,c_3)\right] \left(x_M^{2\,c_3+1} - x_{IN}^{2\,c_3+1}\right),\\
\Phi_3 &= c_1 \left[x_M \left(\ln^2 x_M - 2 \ln x_M + 2\right) - x_{IN} \left(\ln^2 x_{IN} - 2 \ln x_{IN} + 2\right)\right]\\
&\quad - c_1\,[x_M\,(\ln x_M - 1) - x_{IN}\,(\ln x_{IN} - 1)] + \frac{c_2 - 2\,c_1}{4}\,(x_M - x_{IN}),\\
\Phi_4 &= \frac{c_3\,(c_1 - c_2)}{c_3 + 2} \left[x_M^{c_3+2} \left(\ln x_M - \frac{1}{c_3 + 2}\right) - x_{IN}^{c_3+2} \left(\ln x_{IN} - \frac{1}{c_3 + 2}\right)\right]\\
&\quad + \frac{1}{c_3 + 2} \left[2\,c_1 + \frac{c_3\,(c_2 - 7c_1)}{3}\right] \left(x_M^{c_3+2} - x_{IN}^{c_3+2}\right),\\
\Phi_5 &= \frac{3\,c_1 - c_2}{2} \left[x_M^2 \left(\ln x_M - \frac{1}{2}\right) - x_{IN}^2 \left(\ln x_{IN} - \frac{1}{2}\right)\right] - \frac{4\,c_1 - c_2}{6} \left(x_M^2 - x_{IN}^2\right),\\
\Phi_6 &= \frac{2\,c_1\,(1 - c_3)}{c_3 + 1} \left[x_M^{c_3+1} \left(\ln x_M - \frac{1}{c_3 + 1}\right) - x_{IN}^{c_3+1} \left(\ln x_{IN} - \frac{1}{c_3 + 1}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{c_2 c_3 - c_1}{c_3+1}\left(x_M^{c_3+1} - x_{IN}^{c_3+1}\right),\\
\Psi_1 &= \frac{x_M^3}{3}\left[\left(\ln x_M - 3\right)\left(\ln x_M - \frac{1}{3}\right) + \frac{17}{9}\right] - \frac{x_{IN}^3}{3}\left[\left(\ln x_{IN} - 3\right)\left(\ln x_{IN} - \frac{1}{3}\right) + \frac{17}{9}\right],\\
\Psi_2 &= \frac{x_M^{2c_3+1} - x_{IN}^{2c_3+1}}{2c_3+1},\\
\Psi_3 &= x_M \ln x_M \left(\ln x_M - 1\right) - x_{IN} \ln x_{IN}\left(\ln x_{IN} - 1\right) + \frac{5\left(x_M - x_{IN}\right)}{4},\\
\Psi_4 &= \frac{2}{c_3+2}\left\{x_M^{c_3+2}\left[\frac{4c_3+11}{3\left(c_3+2\right)} - \ln x_M\right] - x_{IN}^{c_3+2}\left[\frac{4c_3+11}{3\left(c_3+2\right)} - \ln x_{IN}\right]\right\},\\
\Psi_5 &= \frac{2\left(x_M^2 - x_{IN}^2\right)}{3} + \frac{5}{6}\left[x_M^2\left(\ln x_M - \frac{1}{2}\right) - x_{IN}^2\left(\ln x_{IN} - \frac{1}{2}\right)\right]\\
&- x_M^2\left(\ln^2 x_M - \ln x_M + \frac{1}{2}\right) + x_{IN}^2\left(\ln^2 x_{IN} - \ln x_{IN} + \frac{1}{2}\right),\\
\Psi_6 &= \frac{2}{c_3+1}\left[x_M^{c_3+1}\left(\ln x_M - \frac{1}{c_3+1}\right) - x_{IN}^{c_3+1}\left(\ln x_{IN} - \frac{1}{c_3+1}\right)\right]\\
&+ \frac{1}{c_3+1}\left(x_M^{c_3+1} - x_{IN}^{c_3+1}\right),
\end{aligned}
\tag{5.14}
$$

where x_{IN}, x_M are given by Equations (1.16), (1.17), respectively. The integrals (4.14), which consider Equation (5.14), are determined by the formulae in Chapter 10 (see Equations (10.10)–(10.12)).

In case of the ellipsoidal inclusion, we get $(u_{nIN})_{x_n \to 0} \longrightarrow \pm\infty$, $(\varepsilon_{IN})_{x_n \longrightarrow 0} \longrightarrow \pm\infty$, $(\sigma_{IN})_{r \to 0} \longrightarrow \pm\infty$ due to $(\ln x_n)_{x_n \to 0} \longrightarrow \pm\infty$ and $(x_n^{c_3})_{x_n \to 0} \longrightarrow \pm\infty$ for $c_3 < 0$ (see Equations (2.18), (5.4)–(5.10)). Accordingly, the mathematical solutions (5.4)–(5.10) are suitable for the matrix.

5.2 Matrix

The integration constants C_{1M}, C_{2M}, C_{3M} for the matrix (see Equation (4.4)) are determined by the boundary conditions (2.30), (2.31) or (2.30)–(2.32). The boundary conditions result in the following combinations of C_{1M}, C_{2M}, C_{3M}. Finally, such a combination is considered to exhibit a minimum value of the elastic energy W_C of the cubic cell (see Equation (2.27)).

Conditions $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} = 0$. With regard to Equations (2.30), (2.31), (5.4)–(5.12), (2.21), (2.26), (2.27), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta}\left[\frac{1}{3} - \ln x_n - c_{3M}\left(\frac{4}{3} - \ln x_M\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right], \tag{5.15}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta}\left[\frac{4}{3} - \ln x_n - \left(\frac{4}{3} - \ln x_M\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right], \tag{5.16}$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = \left(\ln x_n - \frac{4}{3}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right) + x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\left(\frac{4}{3} - \ln x_M\right)\right], \tag{5.17}$$

$$\begin{aligned}\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = \Theta\Bigg\{&\left(\ln x_n - \frac{4}{3}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\\ &+ x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\left(\frac{4}{3} - \ln x_M\right)\right]\Bigg\},\end{aligned} \tag{5.18}$$

$$\begin{aligned}\sigma_{nM} = \frac{p_n}{\zeta}\Bigg\{&\frac{7c_{2M} - c_{1M}}{3} + (c_{1M} - c_{2M})\ln x_n\\ &+ [(c_{1M} + c_{2M})c_{3M} - 2c_{2M}]\left(\frac{4}{3} - \ln x_M\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\Bigg\},\end{aligned} \tag{5.19}$$

$$\sigma_{1M} = \eta_{1M} + \eta_{2M}\ln x_n + \eta_{3M}\, x_n^{c_{3M}-1}, \tag{5.20}$$

$$\begin{aligned}\sigma_{\varphi M} = \sigma_{\theta M} = \frac{p_n}{\zeta}\Bigg[&\frac{c_{2M} - 4c_{1M}}{3} + (c_{1M} - c_{2M})\ln x_n\\ &+ (c_{1M} - c_{2M}c_{3M})\left(\frac{4}{3} - \ln x_M\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\Bigg],\end{aligned} \tag{5.21}$$

$$\begin{aligned}w_M = \left(\frac{p_n}{\zeta}\right)^2&\left\{\kappa_{1M} + \kappa_{2M}\left(\frac{4 - 3\ln x_M}{3x_M^{c_{3M}-1}}\right)^2 + \frac{\kappa_{4M}(3\ln x_M - 4)}{3x_M^{c_{3M}-1}}\right\}\\ &+ \frac{\chi_{1M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right]^2\right\}\end{aligned}$$

$$+\frac{\chi_{2M}}{s_{44M}}\left(\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n\,(4-3\ln x_M)}{3\,\zeta\,x_M^{c_{3M}-1}}\right]\right\}^2+\Theta^2\left\{\frac{\partial}{\partial\nu}\left[\frac{p_n\,(4-3\ln x_M)}{3\,\zeta\,x_M^{c_{3M}-1}}\right]\right\}^2\right)$$

$$+\frac{\chi_{4M}}{s_{44M}}\,\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n\,(3\ln x_M-4)}{3\,\zeta\,x_M^{c_{3M}-1}}\right]$$

$$+\frac{\chi_{4M}\,\Theta^2}{s_{44M}}\,\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\nu}\left[\frac{p_n\,(3\ln x_M-4)}{3\,\zeta\,x_M^{c_{3M}-1}}\right], \qquad (5.22)$$

$$W_M=4\int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{p_n}{\zeta}\right)^2\left\{\Phi_{1M}+\Phi_{2M}\left(\frac{4-3\ln x_M}{3x_M^{c_{3M}-1}}\right)^2+\frac{\Phi_{4M}\,(3\ln x_M-4)}{3x_M^{c_{3M}-1}}\right\}d\varphi\,d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{1M}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right]^2\right\}d\varphi\,d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{2M}\left(\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n\,(4-3\ln x_M)}{3\,\zeta\,x_M^{c_{3M}-1}}\right]\right\}^2\right)d\varphi\,d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{2M}\,\Theta^2\left(\left\{\frac{\partial}{\partial\nu}\left[\frac{p_n\,(4-3\ln x_M)}{3\,\zeta\,x_M^{c_{3M}-1}}\right]\right\}^2\right)d\varphi\,d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{4M}\,\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n\,(3\ln x_M-4)}{3\,\zeta\,x_M^{c_{3M}-1}}\right]d\varphi\,d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{4M}\,\Theta^2\,\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\nu}\left[\frac{p_n\,(3\ln x_M-4)}{3\,\zeta\,x_M^{c_{3M}-1}}\right]d\varphi\,d\nu, \qquad (5.23)$$

where Θ, x_M, s_{44M}, c_{iM} ($i=1,2,3$) and κ_{jM}, χ_{jM}; Φ_{jM}, Ψ_{jM} ($j=1,2,4$) are given by Equations (1.15), (1.17), (2.13), (2.18) and (4.13); (4.15), respectively, and ζ, ζ_i ($i=1,2$), η_{jM} ($j=1,2,3$; see Equation (5.13)) have the forms

$$\zeta=\zeta_2-\zeta_1\left(\frac{4}{3}-\ln x_M\right),\quad \zeta_1=\left[(c_{1M}+c_{2M})\,c_{3M}-2\,c_{2M}\right]\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1},$$

$$\zeta_2=\frac{c_{1M}-7c_{2M}}{3}-(c_{1M}-c_{2M})\ln x_{IN},$$

$$\eta_{1M}=-\frac{1}{3}\left\{\frac{p_n\,(\gamma_{1M}+4\gamma_{2M})}{\zeta}+\frac{4}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)+\gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right]\right\},$$

$$\eta_{2M} = \frac{p_n\left(\gamma_{1M}+\gamma_{2M}\right)}{\zeta} + \frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right) + \gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right],$$

$$\eta_{3M} = C_2\left(\gamma_{1M}c_{3M}+\gamma_{2M}\right) + \frac{\gamma_{3M}}{s_{44M}}\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\left(\frac{4}{3}-\ln x_M\right)\right]$$
$$+ \frac{\gamma_{4M}}{s_{44M}}\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\left(\frac{4}{3}-\ln x_M\right)\right], \tag{5.24}$$

The normal stress p_n is given by Equation (2.33). With regard to Equation (5.16), the coefficient ρ_M in Equation (2.33) is derived as

$$\rho_M = \frac{1}{\zeta}\left[\frac{4}{3}-\ln x_{IN} - \left(\frac{4}{3}-\ln x_M\right)\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1}\right]. \tag{5.25}$$

Conditions $C_{1M}\neq 0$, $C_{3M}\neq 0$, $C_{2M}=0$. With regard to Equations (2.30), (2.31), (5.4)–(5.12), (2.21), (2.26), (2.27), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta}\left[\left(\frac{1}{2}+\ln x_M\right)\left(\frac{1}{3}-\ln x_n\right) - \frac{x_M}{x_n}\left(\frac{4}{3}-\ln x_M\right)\right], \tag{5.26}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta}\left[\left(\frac{1}{2}+\ln x_M\right)\left(\frac{4}{3}-\ln x_n\right) - \frac{x_M}{x_n}\left(\frac{4}{3}-\ln x_M\right)\left(\frac{1}{2}+\ln x_n\right)\right], \tag{5.27}$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = \left(\ln x_n - \frac{4}{3}\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]$$
$$+ \frac{1}{x_n}\left(\frac{1}{2}+\ln x_n\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n x_M}{\zeta}\left(\ln x_M - \frac{4}{3}\right)\right], \tag{5.28}$$

$$\varepsilon_{n\nu M} = s_{44M}\,\sigma_{n\varphi M} = \Theta\left\{\left(\ln x_n - \frac{4}{3}\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\right.$$
$$\left.+ \frac{1}{x_n}\left(\frac{1}{2}+\ln x_n\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n x_M}{\zeta}\left(\ln x_M - \frac{4}{3}\right)\right]\right\}, \tag{5.29}$$

$$\sigma_{nM} = \frac{p_n}{\zeta}\left\{\left(\frac{1}{2}+\ln x_M\right)\left[\frac{c_{1M}-7c_{2M}}{3} - \left(c_{1M}-c_{2M}\right)\ln x_n\right]\right.$$
$$\left.+ \frac{x_M}{x_n}\left(\frac{4}{3}-\ln x_M\right)\left(c_{1M}-2c_{2M}\ln x_n\right)\right\}, \tag{5.30}$$

$$\sigma_{1M} = \eta_{1M} + \eta_{2M} \ln x_n + \frac{\eta_{4M} + \eta_{5M} \ln x_n}{x_n}, \tag{5.31}$$

$$\begin{aligned} \sigma_{\varphi M} = \sigma_{\theta M} = \frac{p_n}{\zeta} \Bigg\{ \left(\frac{1}{2} + \ln x_M\right) \left[\frac{4c_{1M} - c_{2M}}{3} - (c_{1M} - c_{2M}) \ln x_n\right] \\ + \frac{x_M}{x_n} \left(\frac{4}{3} - \ln x_M\right) \left(\frac{c_{1M} - c_{2M}}{2} + c_{3M} \ln x_n\right) \Bigg\}, \end{aligned} \tag{5.32}$$

$$\begin{aligned} w_M = \left(\frac{p_n}{\zeta}\right)^2 \Bigg[\kappa_{1M} \left(\frac{1}{2} + \ln x_M\right) + \kappa_{3M} x_M^2 \left(\frac{4}{3} - \ln x_M\right)^2 \\ + \kappa_{5M} x_M \left(\frac{1}{2} + \ln x_M\right) \left(\frac{4}{3} - \ln x_M\right) \Bigg] \\ + \frac{\chi_{1M}}{s_{44M}} \left(\left\{ \frac{\partial}{\partial \varphi} \left[\frac{p_n}{\zeta} \left(\frac{1}{2} + \ln x_M\right) \right] \right\}^2 + \Theta^2 \left\{ \frac{\partial}{\partial \nu} \left[\frac{p_n}{\zeta} \left(\frac{1}{2} + \ln x_M\right) \right] \right\}^2 \right) \\ + \frac{\chi_{3M}}{s_{44M}} \left(\left\{ \frac{\partial}{\partial \varphi} \left[\frac{p_n x_M}{\zeta} \left(\frac{4}{3} - \ln x_M\right) \right] \right\}^2 + \Theta^2 \left\{ \frac{\partial}{\partial \nu} \left[\frac{p_n x_M}{\zeta} \left(\frac{4}{3} - \ln x_M\right) \right] \right\}^2 \right) \\ + \frac{\chi_{5M}}{s_{44M}} \frac{\partial}{\partial \varphi} \left[\frac{p_n}{\zeta} \left(\frac{1}{2} + \ln x_M\right) \right] \frac{\partial}{\partial \varphi} \left[\frac{p_n x_M}{\zeta} \left(\frac{4}{3} - \ln x_M\right) \right] \\ + \frac{\chi_{5M} \Theta^2}{s_{44M}} \frac{\partial}{\partial \nu} \left[\frac{p_n}{\zeta} \left(\frac{1}{2} + \ln x_M\right) \right] \frac{\partial}{\partial \nu} \left[\frac{p_n x_M}{\zeta} \left(\frac{4}{3} - \ln x_M\right) \right], \end{aligned} \tag{5.33}$$

$$\begin{aligned} W_M = 4 \int_0^{\pi/2} \int_0^{\pi/2} \left(\frac{p_n}{\zeta}\right)^2 \Bigg[\Phi_{1M} \left(\frac{1}{2} + \ln x_M\right) + \Phi_{3M} x_M^2 \left(\frac{4}{3} - \ln x_M\right)^2 \\ + \Phi_{5M} x_M \left(\frac{1}{2} + \ln x_M\right) \left(\frac{4}{3} - \ln x_M\right) \Bigg] \, d\varphi \, d\nu \\ + \frac{4}{s_{44M}} \int_0^{\pi/2} \int_0^{\pi/2} \Psi_{1M} \left(\left\{ \frac{\partial}{\partial \varphi} \left[\frac{p_n}{\zeta} \left(\frac{1}{2} + \ln x_M\right) \right] \right\}^2 \right. \\ \left. + \Theta^2 \left\{ \frac{\partial}{\partial \nu} \left[\frac{p_n}{\zeta} \left(\frac{1}{2} + \ln x_M\right) \right] \right\}^2 \right) \, d\varphi \, d\nu \\ + \frac{4}{s_{44M}} \int_0^{\pi/2} \int_0^{\pi/2} \left(\Psi_{3M} \left\{ \frac{\partial}{\partial \varphi} \left[\frac{p_n x_M}{\zeta} \left(\frac{4}{3} - \ln x_M\right) \right] \right\}^2 \right. \end{aligned}$$

$$
+\,\Theta^2\left\{\frac{\partial}{\partial\nu}\left[\frac{p_n x_M}{\zeta}\left(\frac{4}{3}-\ln x_M\right)\right]\right\}^2\Bigg)\,d\varphi\,d\nu
$$

$$
+\,\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{5M}\,\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\frac{\partial}{\partial\varphi}\left[\frac{p_n x_M}{\zeta}\left(\frac{4}{3}-\ln x_M\right)\right]d\varphi\,d\nu
$$

$$
+\,\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Phi_{5M}\,\Theta^2\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\frac{\partial}{\partial\nu}\left[\frac{p_n x_M}{\zeta}\left(\frac{4}{3}-\ln x_M\right)\right]d\varphi\,d\nu, \tag{5.34}
$$

where Θ, x_M, s_{44M}, c_{iM} $(i=1,2,3)$ and κ_{jM}, χ_{jM}; Φ_{jM}, Ψ_{jM} $(j=1,3,5)$ are given by Equations (1.15), (1.17), (2.13), (2.18) and (4.13); (4.15), respectively, and ζ, ζ_i $(i=1,2)$, η_{jM} ($j=1,2,4,5$; see Equation (5.13)) have the forms

$$
\begin{aligned}
\zeta &= \frac{\zeta_2}{x_M}\left(\frac{1}{2}+\ln x_M\right)-\zeta_1\left(\frac{4}{3}-\ln x_M\right), \quad \zeta_1=\frac{x_M}{x_{IN}}\left(c_{1M}-2c_{2M}\ln x_{IN}\right),\\
\zeta_2 &= x_M\left[\frac{c_{1M}-7c_{2M}}{3}-\left(c_{1M}-c_{2M}\right)\ln x_{IN}\right],\\
\eta_{1M} &= -\frac{p_n\left(\gamma_{1M}+4\gamma_{2M}\right)}{3\zeta}\left(\frac{1}{2}+\ln x_M\right)-\frac{4\gamma_{3M}}{3s_{44M}}\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\\
&\quad-\frac{4\gamma_{4M}}{3s_{44M}}\frac{\partial C_1}{\partial\nu}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right],\\
\eta_{2M} &= \frac{p_n\left(\gamma_{1M}+\gamma_{2M}\right)}{\zeta}\left(\frac{1}{2}+\ln x_M\right)+\frac{\gamma_{3M}}{s_{44M}}\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\\
&\quad+\frac{\gamma_{4M}}{s_{44M}}\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right],\\
\eta_{4M} &= \frac{p_n x_M\left(2\gamma_{1M}+\gamma_{2M}\right)}{2\zeta}\left(\ln x_M-\frac{4}{3}\right)+\frac{\gamma_{3M}}{2s_{44M}}\frac{\partial}{\partial\varphi}\left[\frac{p_n x_M}{\zeta}\left(\ln x_M-\frac{4}{3}\right)\right]\\
&\quad+\frac{\gamma_{4M}}{2s_{44M}}\frac{\partial}{\partial\nu}\left[\frac{p_n x_M}{\zeta}\left(\ln x_M-\frac{4}{3}\right)\right],\\
\eta_{5M} &= \frac{p_n x_M\gamma_{2M}}{\zeta}\left(\ln x_M-\frac{4}{3}\right)+\frac{\gamma_{3M}}{s_{44M}}\frac{\partial}{\partial\varphi}\left[\frac{p_n x_M}{\zeta}\left(\ln x_M-\frac{4}{3}\right)\right]\\
&\quad+\frac{\gamma_{4M}}{s_{44M}}\frac{\partial}{\partial\nu}\left[\frac{p_n x_M}{\zeta}\left(\ln x_M-\frac{4}{3}\right)\right].
\end{aligned} \tag{5.35}
$$

The normal stress p_n is given by Equation (2.33). With regard to Equation (5.27), the coefficient ρ_M in Equation (2.33) is derived as

$$\rho_M = \frac{1}{\zeta}\left[\left(\frac{1}{2}+\ln x_M\right)\left(\frac{4}{3}-\ln x_{IN}\right) - \frac{x_M}{x_{IN}}\left(\frac{4}{3}-\ln x_M\right)\left(\frac{1}{2}+\ln x_{IN}\right)\right]. \quad (5.36)$$

Conditions $C_{2M} \neq 0$, $C_{3M} \neq 0$, $C_{1M} = 0$. With regard to Equations (2.30), (2.31), (5.4)–(5.12), (2.21), (2.26), (2.27), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta}\left[c_{3M}\left(\frac{1}{2}+\ln x_M\right)x_n^{c_{3M}-1} - \frac{x_M^{c_{3M}}}{x_n}\right], \quad (5.37)$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta}\left[\left(\frac{1}{2}+\ln x_M\right)x_n^{c_{3M}-1} - \frac{x_M^{c_{3M}}}{x_n}\left(\frac{1}{2}+\ln x_n\right)\right], \quad (5.38)$$

$$\varepsilon_{n\varphi M} = s_{44M}\sigma_{n\varphi M} = \frac{1}{x_n}\left(\frac{1}{2}+\ln x_n\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^{c_{3M}}}{\zeta}\right) - x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right], \quad (5.39)$$

$$\varepsilon_{n\nu M} = s_{44M}\sigma_{n\nu M} = \Theta\left\{\frac{1}{x_n}\left(\frac{1}{2}+\ln x_n\right)\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^{c_{3M}}}{\zeta}\right) - x_n^{c_{3M}-1}\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\right\}, \quad (5.40)$$

$$\sigma_{nM} = -\frac{p_n x_n^{c_{3M}-1}}{\zeta} \times\left\{\left[c_{3M}(c_{1M}+c_{2M})-2c_{2M}\right]\left(\frac{1}{2}+\ln x_M\right) - \frac{x_M}{x_n}(c_{1M}-2c_{2M}\ln x_n)\right\}, \quad (5.41)$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n x_n^{c_{3M}-1}}{\zeta}\left[(c_{1M}-c_{2M}c_{3M})\left(\frac{1}{2}+\ln x_M\right) - \frac{x_M}{x_n}\left(\frac{c_{1M}-2c_{2M}}{2}+c_{1M}\ln x_n\right)\right], \quad (5.42)$$

$$\sigma_{1M} = \eta_{3M}x_n^{c_{3M}-1} + \frac{\eta_{4M}+\eta_{5M}\ln x_n}{x_n}, \quad (5.43)$$

$$w_M = \left(\frac{p_n}{\zeta}\right)^2\left[\kappa_{2M}\left(\frac{1}{2}+\ln x_M\right)^2 + \kappa_{3M}x_M^{2c_{3M}} + \kappa_{6M}x_M^{c_{3M}}\left(\frac{1}{2}+\ln x_M\right)\right]$$

$$+\frac{\chi_{2M}}{s_{44M}}\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]^2+\Theta^2\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]^2\right\}$$

$$+\frac{\chi_{3M}}{s_{44M}}\left[\frac{\partial C_{3M}}{\partial\varphi}\left(\frac{p_n x_M^{c_{3M}}}{\zeta}\right)^2+\Theta^2\frac{\partial C_{3M}}{\partial\nu}\left(\frac{p_n x_M^{c_{3M}}}{\zeta}\right)^2\right]$$

$$-\frac{\chi_{6M}\Theta}{s_{44M}}\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^{c_{3M}}}{\zeta}\right)\right.$$

$$\left.+\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^{c_{3M}}}{\zeta}\right)\right\}, \quad (5.44)$$

$$W_M=4\int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{p_n}{\zeta}\right)^2\left[\Phi_{2M}\left(\frac{1}{2}+\ln x_M\right)^2+\Phi_{3M}x_M^{2c_{3M}}\right.$$

$$\left.+\Phi_{6M}x_M^{c_{3M}}\left(\frac{1}{2}+\ln x_M\right)\right]d\varphi\,d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{2M}\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]^2+\Theta^2\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]^2\right\}d\varphi\,d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{3M}\left[\frac{\partial C_{3M}}{\partial\varphi}\left(\frac{p_n x_M^{c_{3M}}}{\zeta}\right)^2+\Theta^2\frac{\partial C_{3M}}{\partial\nu}\left(\frac{p_n x_M^{c_{3M}}}{\zeta}\right)^2\right]d\varphi\,d\nu$$

$$-\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{6M}\Theta\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^{c_{3M}}}{\zeta}\right)\right.$$

$$\left.+\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^{c_{3M}}}{\zeta}\right)\right\}d\varphi\,d\nu, \quad (5.45)$$

where Θ, x_M, s_{44M}, c_{iM} ($i=1,2,3$) and κ_{jM}, χ_{jM}; Φ_{jM}, Ψ_{jM} ($j=1,3,5$) are given by Equations (1.15), (1.17), (2.13), (2.18) and (4.13); (4.15), respectively, and ζ, ζ_i ($i=1,2$), η_{jM} ($j=3,4,5$; see Equation (5.13)) have the forms

$$\zeta=\frac{\zeta_2}{x_M}\left(\frac{1}{2}+\ln x_M\right)-\zeta_1 x_M^{c_{3M}-1},\quad \zeta_1=\frac{x_M}{x_{IN}}\left(c_{1M}-2c_{2M}\ln x_{IN}\right),$$

$$\zeta_2=x_M\left[c_{3M}\left(c_{1M}+c_{2M}\right)-2c_{2M}\right]x_{IN}^{c_{3M}-1},$$

$$\eta_{3M}=\frac{p_n x_M^{c_{3M}}\left(\gamma_{1M}c_{3M}+\gamma_{2M}\right)}{\zeta}+\frac{\gamma_{3M}}{s_{44M}}\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^{c_{3M}}}{\zeta}\right)$$

$$+\frac{\gamma_{4M}}{s_{44M}}\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^{c_{3M}}}{\zeta}\right),$$

$$\eta_{4M} = \frac{p_n (2\gamma_{1M} + \gamma_{2M})}{2\zeta}\left(\frac{1}{2} + \ln x_M\right) + \frac{\gamma_{3M}}{2 s_{44M}}\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2} + \ln x_M\right)\right]$$
$$+ \frac{\gamma_{4M}}{2 s_{44M}}\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta}\left(\frac{1}{2} + \ln x_M\right)\right],$$
$$\eta_{5M} = \frac{p_n \gamma_{2M}}{\zeta}\left(\frac{1}{2} + \ln x_M\right) + \frac{\gamma_{3M}}{s_{44M}}\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2} + \ln x_M\right)\right]$$
$$+ \frac{\gamma_{4M}}{s_{44M}}\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta}\left(\frac{1}{2} + \ln x_M\right)\right]. \quad (5.46)$$

The normal stress p_n is given by Equation (2.33). With regard to Equation (5.38), the coefficient ρ_M in Equation (2.33) is derived as

$$\rho_M = \frac{1}{\zeta}\left[\left(\frac{1}{2} + \ln x_M\right) x_{IN}^{c_{3M}-1} - \frac{x_M^{c_{3M}}}{x_{IN}}\left(\frac{1}{2} + \ln x_{IN}\right)\right]. \quad (5.47)$$

Conditions $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} \neq 0$. With regard to Equations (2.30)–(2.32), (5.4)–(5.12), (2.21), (2.26), (2.27), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta}\left[\zeta_1\left(\frac{1}{3} - \ln x_n\right) + \zeta_2 c_{3M} x_n^{c_{3M}-1} + \frac{\zeta_3}{x_n}\right], \quad (5.48)$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta}\left[\zeta_1\left(\frac{4}{3} - \ln x_n\right) + \zeta_2 x_n^{c_{3M}-1} + \frac{\zeta_{3mI}}{x_n}\left(\frac{1}{2} + \ln x_n\right)\right], \quad (5.49)$$

$$\varepsilon_{n\varphi M} = s_{44M}\sigma_{n\varphi M} = -\left[\left(\frac{4}{3} - \ln x_n\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right) + x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right)\right.$$
$$\left.+ \frac{1}{x_n}\left(\frac{1}{2} + \ln x_n\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right)\right], \quad (5.50)$$

$$\varepsilon_{n\nu M} = s_{44M}\sigma_{n\varphi M} = -\Theta\left[\left(\frac{4}{3} - \ln x_n\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right) + x_n^{c_{3M}-1}\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right)\right.$$
$$\left.+ \frac{1}{x_n}\left(\frac{1}{2} + \ln x_n\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right], \quad (5.51)$$

$$\sigma_{nM} = -\frac{p_n}{\zeta}\left\{\zeta_1\left[\frac{c_{1M} - 7c_{2M}}{3} - (c_{1M} - c_{2M})\ln x_n\right]\right.$$
$$\left.+ \zeta_2\left[(c_{1M} + c_{2M})c_{3M} - 2c_{2M}\right]x_n^{c_{3M}-1} + \frac{\zeta_{3mI}(c_{1M} - 2c_{2M}\ln x_n)}{x_n}\right\}, \quad (5.52)$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta}\left\{\zeta_1\left[\frac{4c_{1M}-c_{2M}}{3} - (c_{1M}-c_{2M})\ln x_n\right]\right.$$
$$\left. + \zeta_2 (c_{1M} - c_{2M} c_{3M}) x_n^{c_{3M}-1} + \zeta_{3mI}\left(\frac{c_{1M}-2c_{2M}}{2} + c_{1M}\ln x_n\right)\right\}, \tag{5.53}$$

$$\sigma_{1M} = \eta_{1M} + \eta_{2M}\ln x_n + \eta_{3M} x_n^{c_{3M}-1} + \frac{\eta_{4M} + \eta_{5M}\ln x_n}{x_n}, \tag{5.54}$$

$$w_M = \left(\frac{p_n}{\zeta}\right)^2 \left(\kappa_{1M}\zeta_1^2 + \kappa_{2M}\zeta_2^2 + \kappa_{3M}\zeta_3^2 + \kappa_{4M}\zeta_1\zeta_2 + \kappa_{5M}\zeta_1\zeta_3 + \kappa_{6M}\zeta_2\zeta_3\right)$$
$$+ \frac{\chi_{1M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\right]^2\right\}$$
$$+ \frac{\chi_{2M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right)\right]^2\right\}$$
$$+ \frac{\chi_{3M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]^2\right\}$$
$$+ \frac{\chi_{4M}}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right) + \Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right)\right]$$
$$+ \frac{\chi_{5M}}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right) + \Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]$$
$$+ \frac{\chi_{6M}}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right) + \Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right], \tag{5.55}$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{p_n}{\zeta}\right)^2\left(\Phi_{1M}\zeta_1^2 + \Phi_{2M}\zeta_2^2 + \Phi_{3M}\zeta_3^2\right.$$
$$\left. + \Phi_{4M}\zeta_1\zeta_2 + \Phi_{5M}\zeta_1\zeta_3 + \Phi_{6M}\zeta_2\zeta_3\right) d\varphi\, d\nu$$
$$+ \frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{1M}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\right]^2\right\} d\varphi\, d\nu$$
$$+ \frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{2M}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right)\right]^2\right\} d\varphi\, d\nu$$

$$
\begin{aligned}
&+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{3M}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]^2\right\}d\varphi\, d\nu \\
&+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{4M}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right)\right. \\
&\qquad\qquad\left.+\Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right)\right]d\varphi\, d\nu \\
&+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{5M}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right)\right. \\
&\qquad\qquad\left.+\Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]d\varphi\, d\nu \\
&+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{6M}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right)\right. \\
&\qquad\qquad\left.+\Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]d\varphi\, d\nu, \qquad (5.56)
\end{aligned}
$$

where Θ, x_M, s_{44M}, c_{iM} $(i=1,2,3)$ and κ_{jM}, χ_{jM}; Φ_{jM}, Ψ_{jM} $(j=1,3,5)$ are given by Equations (1.15), (1.17), (2.13), (2.18) and (4.13); (4.15), respectively, and ζ, ζ_i $(i=1,2,3)$, η_{jM} ($j=1,\ldots,5$; see Equation (5.13)) have the forms

$$
\begin{aligned}
\zeta_1 &= x_M^{c_{3M}-1}\left[c_{3M}\left(\frac{1}{2}+\ln x_M\right)-1\right], \\
\zeta_2 &= \frac{4}{3}-\ln x_M-\left(\frac{1}{2}+\ln x_M\right)\left(\frac{1}{3}-\ln x_M\right), \\
\zeta_{3mI} &= x_M^{c_{3M}}\left[\frac{1}{3}-\ln x_M-c_{3M}\left(\frac{4}{3}-\ln x_M\right)\right], \\
\zeta &= x_M^{c_{3M}-1}\left[\frac{c_{1M}-7c_{2M}}{3}-(c_{1M}-c_{2M})\ln x_{IN}\right] \\
&\quad+\left[(c_{1M}+c_{2M})c_{3M}-2c_{2M}\right]\left(\frac{1}{2}+\ln x_M\right)\left(\frac{1}{3}-\ln x_M\right)x_{IN}^{c_{3M}-1} \\
&\quad+\left(c_{1M}-2c_{2M}\ln x_{IN}\right)\left(\frac{4}{3}-\ln x_M\right)\frac{c_{3M}x_M^{c_{3M}}}{x_{IN}} \\
&\quad-\left\{c_{3M}\left[\frac{c_{1M}-7c_{2M}}{3}-(c_{1M}-c_{2M})\ln x_{IN}\right]\left(\frac{1}{2}+\ln x_M\right)x_M^{c_{3M}-1}\right.
\end{aligned}
$$

$$+ \left[(c_{1M}+c_{2M})\,c_{3M}-2\,c_{2M}\right]\left(\frac{4}{3}-\ln x_M\right)x_{IN}^{c_{3M}-1}$$
$$+ \frac{(c_{1M}-2\,c_{2M}\ln x_{IN})\,x_M^{c_{3M}}}{x_{IN}}\left(\frac{1}{3}-\ln x_M\right)\Bigg\},$$
$$\eta_{1M} = -\frac{p_n\zeta_1\,(\gamma_{1M}+4\,\gamma_{2M})}{3\,\zeta}-\frac{4}{3\,s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)+\gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\right],$$
$$\eta_{2M} = \frac{p_n\zeta_1\,(\gamma_{1M}+\gamma_{2M})}{\zeta}+\frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)+\gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\right],$$
$$\eta_{3M} = -\frac{p_n\zeta_2\,(\gamma_{1M}c_{3M}+\gamma_{2M})}{\zeta}-\frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right)+\gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right)\right],$$
$$\eta_{4M} = -\frac{p_n\zeta_3\,(2\,\gamma_{1M}+\gamma_{2M})}{2\,\zeta}-\frac{1}{2\,s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right)+\gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right],$$
$$\eta_{5M} = -\frac{p_n\zeta_3\,\gamma_{2M}}{\zeta}\,\frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right)+\gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]. \qquad (5.57)$$

The normal stress p_n is given by Equation (2.33). With regard to Equation (5.49), the coefficient ρ_M in Equation (2.33) is derived as

$$\rho_M = \frac{1}{\zeta}\left[\zeta_1\left(\frac{4}{3}-\ln x_{IN}\right)+\zeta_2\,x_{IN}^{c_{3M}-1}+\frac{\zeta_{3mI}}{x_{IN}}\left(\frac{1}{2}+\ln x_{IN}\right)\right]. \qquad (5.58)$$

Chapter 6

Mathematical Model 4

6.1 Mathematical Procedure

The differential equation (2.23) is transformed to the form

$$U_n = -s_{44}\left(c_1 + c_2\right)\left(x_n^2 \frac{\partial^2 u_n}{\partial x_n^2} + 2x_n \frac{\partial u_n}{\partial x_n} - 2u_n\right), \tag{6.1}$$

where s_{44}, c_i $(i=1{,}2)$ and $U_n = U_n\left(x_n, \varphi, \nu\right)$ are given by Equations (2.13), (2.18) and (2.25), respectively. Let $x_n\left[\partial \text{Eq.}(6.1)/\partial x_n\right]$ be performed, and then we get

$$x_n \frac{\partial U_n}{\partial x_n} = -s_{44}\left(c_1 + c_2\right)\left(x_n^3 \frac{\partial^3 u_n}{\partial x_n^3} + 4x_n^2 \frac{\partial^2 u_n}{\partial x_n^2}\right). \tag{6.2}$$

Let Equations (6.1), (6.2 be substituted to Equation (2.24, and then we get

$$x_n^3 \frac{\partial^3 u_n}{\partial x_n^3} + \left(4 - c_3\right) x_n^2 \frac{\partial^2 u_n}{\partial x_n^2} - 2c_3 x_n \frac{\partial u_n}{\partial x_n} + 2c_3 u_n = 0. \tag{6.3}$$

Let u_n be assumed in the form $u_n = x_n^{\lambda}$, then we get [1]–[22]

$$u_n = C_1 x_n + C_2 x_n^{c_3} + \frac{C_3}{x_n^2}, \tag{6.4}$$

where $c_3 < 0$ is given by Equation (2.18), and C_1, C_2, C_3 are integration constants, which are determined by the boundary conditions in Section 2.3. With regard to Equations (2.1)–(2.4), (2.14)–(2.17), (2.26), (6.6), we get

$$\varepsilon_n = C_1 + C_2 c_3 x_n^{c_3 - 1} - \frac{2C_3}{x_n^3}, \tag{6.5}$$

$$\varepsilon_\varphi = \varepsilon_\theta = C_1 + C_2 x_n^{c_3 - 1} + \frac{C_3}{x_n^3}, \tag{6.6}$$

$$\varepsilon_{n\varphi} = s_{44}\,\sigma_{n\varphi} = \frac{\partial C_1}{\partial \varphi} + x_n^{c_3 - 1} \frac{\partial C_2}{\partial \varphi} + \frac{1}{x_n^3} \frac{\partial C_3}{\partial \varphi}, \tag{6.7}$$

$$\varepsilon_{n\theta} = s_{44}\,\sigma_{n\theta} = \Theta\left[\frac{\partial C_1}{\partial \nu} + x_n^{c_3 - 1} \frac{\partial C_2}{\partial \nu} + \frac{1}{x_n^3} \frac{\partial C_3}{\partial \nu}\right], \tag{6.8}$$

$$\sigma_n = C_1\,(c_1 - c_2) + C_2\,[(c_1 + c_2)\,c_3 - 2\,c_2]\,x_n^{c_3 - 1} - \frac{2\,C_3\,(c_1 + 2\,c_2)}{x_n^3}, \tag{6.9}$$

$$\sigma_\varphi = \sigma_\theta = C_1\,(c_1 - c_2) + C_2\,(c_1 - c_2\,c_3)\,x_n^{c_3 - 1} + \frac{C_3\,(c_1 + 2\,c_2)}{x_n^3}, \tag{6.10}$$

$$\sigma_1 = \eta_1 + \eta_2\,x_n^{c_3 - 1} + \frac{\eta_3}{x_n^3}, \tag{6.11}$$

$$w = \kappa_1 + \kappa_2\,x_n^{2(c_3 - 1)} + \frac{\kappa_3}{x_n^6} + \kappa_4\,x_n^{c_3 - 1} + \frac{\kappa_5}{x_n^3} + \kappa_6\,x_n^{c_3 - 4}, \tag{6.12}$$

where Θ is given by Equation (1.15), and η, κ_j $(i = 1,2,3;\ j = 1,\ldots,6)$ is derived as

$$\begin{aligned}
\eta_1 &= C_1\,(\gamma_1 + \gamma_2) + \frac{1}{s_{44}}\left(\gamma_3\,\frac{\partial C_1}{\partial \varphi} + \gamma_4\,\frac{\partial C_1}{\partial \nu}\right),\\
\eta_2 &= C_2\,(\gamma_1 c_3 + \gamma_2) + \frac{1}{s_{44}}\left(\gamma_3\,\frac{\partial C_2}{\partial \varphi} + \gamma_4\,\frac{\partial C_2}{\partial \nu}\right),\\
\eta_3 &= C_3\,(\gamma_2 - 2\,\gamma_1) + \frac{1}{s_{44}}\left(\gamma_3\,\frac{\partial C_3}{\partial \varphi} + \gamma_4\,\frac{\partial C_3}{\partial \nu}\right),\\
\kappa_1 &= \frac{3\,(c_1 - c_2)\,C_1^2}{2} + \frac{1}{s_{44}}\left[\left(\frac{\partial C_1}{\partial \varphi}\right)^2 + \Theta^2\left(\frac{\partial C_1}{\partial \nu}\right)^2\right],\\
\kappa_2 &= \left[\frac{(c_1 + c_2)\,c_3^2}{2} + c_1 - 2\,c_2\,c_3\right] C_2^2 + \frac{1}{s_{44}}\left[\left(\frac{\partial C_2}{\partial \varphi}\right)^2 + \Theta^2\left(\frac{\partial C_2}{\partial \nu}\right)^2\right],\\
\kappa_3 &= 3\,(c_1 + 2\,c_2)\,C_3^2 + \frac{1}{s_{44}}\left[\left(\frac{\partial C_3}{\partial \varphi}\right)^2 + \Theta^2\left(\frac{\partial C_3}{\partial \nu}\right)^2\right],\\
\kappa_4 &= (c_1 - c_2)\,(2 + c_3)\,C_1\,C_2 + \frac{2}{s_{44}}\left(\frac{\partial C_1}{\partial \varphi}\,\frac{\partial C_2}{\partial \varphi} + \Theta^2\,\frac{\partial C_1}{\partial \nu}\,\frac{\partial C_2}{\partial \nu}\right),\\
\kappa_5 &= \frac{2}{s_{44}}\left(\frac{\partial C_1}{\partial \varphi}\,\frac{\partial C_3}{\partial \varphi} + \Theta^2\,\frac{\partial C_1}{\partial \nu}\,\frac{\partial C_3}{\partial \nu}\right),\\
\kappa_6 &= [2\,c_2\,(1 - c_3) - c_1]\,C_2\,C_3 + \frac{2}{s_{44}}\left(\frac{\partial C_2}{\partial \varphi}\,\frac{\partial C_3}{\partial \varphi} + \Theta^2\,\frac{\partial C_2}{\partial \nu}\,\frac{\partial C_3}{\partial \nu}\right).
\end{aligned} \tag{6.13}$$

6.2 Matrix

The integration constants C_{1M}, C_{2M}, C_{3M} for the matrix (see Equation (4.4)) are determined by the boundary conditions (2.30), (2.31) or (2.30)–(2.32). The boundary conditions result in the following combinations of C_{1M}, C_{2M}, C_{3M}. Finally, such a combination is considered to exhibit a minimum value of the elastic energy W_C of the cubic cell (see Equation (2.27)).

Conditions $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} = 0$. With regard to Equations (2.30), (2.31), (6.4)–(6.12), (2.21), (2.26), (2.27), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta}\left[1 - c_{3M}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right], \tag{6.14}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta}\left[1 - \left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right], \tag{6.15}$$

$$\varepsilon'_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = -\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right) - x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\right], \tag{6.16}$$

$$\varepsilon'_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = -\Theta\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right) - x_n^{c_{3M}-1}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\right], \tag{6.17}$$

$$\sigma_{nM} = -\frac{p_n}{\zeta}\left\{c_{1M} - c_{2M} - \left[c_{3M}\left(c_{1M} + c_{2M}\right) - 2c_{2M}\right]\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right\}, \tag{6.18}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta}\left[c_{1M} - c_{2M} - \left(c_{1M} - c_{2M}c_{3M}\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right], \tag{6.19}$$

$$\sigma_{1M} = \eta_{1M} + \eta_{2M}\,x_n^{c_{3M}-1}, \tag{6.20}$$

$$w_M = \kappa_{1M} + \kappa_{2M}\,x_n^{2(c_{3M}-1)} + \kappa_{4M}\,x_n^{c_{3M}-1}, \tag{6.21}$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\kappa_{1M}}{3}\left(x_M^3 - x_{IN}^3\right) + \frac{\kappa_{2M}}{2c_{3M}+1}\left(x_M^{2c_{3M}+1} - x_{IN}^{2c_{3M}+1}\right)\right.$$
$$\left. + \frac{\kappa_{4M}}{c_{3M}+2}\left(x_M^{c_{3M}+2} - x_{IN}^{c_{3M}+2}\right)\right] d\varphi\, d\nu, \tag{6.22}$$

where Θ, x_{IN}, x_M and s_{44M}, c_{iM} $(i = 1,2,3)$ are given by Equations (1.15)–(1.17) and (2.13), (2.18), respectively, and ζ, η_{iM}, κ_{jM} $(i = 1,2;\ j = 1,2,4$; see Equation (6.13)) have the forms

$$\zeta = c_{1M} - c_{2M} - [c_{3M}(c_{1M} + c_{2M}) - 2c_{2M}]\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1},$$

$$\eta_{1M} = -\frac{p_n(\gamma_{1M} + \gamma_{2M})}{\zeta} - \frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right) + \gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right],$$

$$\eta_{2M} = \frac{p_n(\gamma_{1M}c_{3M} + \gamma_{2M})}{\zeta x_M^{c_{3M}-1}} + \frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta x_M^{c_{3M}-1}}\right) + \gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta x_M^{c_{3M}-1}}\right)\right],$$

$$\kappa_{1M} = \frac{3(c_{1M} - c_{2M})}{2}\left(\frac{p_n}{\zeta}\right)^2 + \frac{1}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right]^2\right\},$$

$$\kappa_{2M} = \left[\frac{(c_{1M} + c_{2M})c_{3M}^2}{2} + c_{1M} - 2c_{2M}c_{3M}\right]\left(\frac{p_n}{\zeta x_M^{c_{3M}-1}}\right)^2 + \frac{1}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta x_M^{c_{3M}-1}}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta x_M^{c_{3M}-1}}\right)\right]^2\right\},$$

$$\kappa_{4M} = \frac{(c_{2M} - c_{1M})(2 + c_{3M})}{x_M^{c_{3M}-1}}\left(\frac{p_n}{\zeta}\right)^2 - \frac{2}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta x_M^{c_{3M}-1}}\right) + \Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta x_M^{c_{3M}-1}}\right)\right]. \tag{6.23}$$

The normal stress p_n is given by Equation (2.33). With regard to Equation (6.15), the coefficient ρ_M in Equation (2.33) is derived as

$$\rho_M = \frac{1}{\zeta}\left[1 - \left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1}\right]. \tag{6.24}$$

Conditions $C_{1M} \neq 0$, $C_{3M} \neq 0$, $C_{2M} = 0$. With regard to Equations (2.30), (2.31), (6.5)–(6.12), (2.21), (2.26), (2.27), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta}\left[1 - c_{3M}\left(\frac{x_M}{x_n}\right)^3\right], \tag{6.25}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta}\left[1 - \left(\frac{x_M}{x_n}\right)^3\right], \tag{6.26}$$

$$\varepsilon_{n\varphi M} = s_{44M}\sigma_{n\varphi M} = -\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right) - \frac{1}{x_n^3}\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{\zeta}\right)\right], \tag{6.27}$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = -\Theta\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right) - \frac{1}{x_n^3}\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^3}{\zeta}\right)\right], \tag{6.28}$$

$$\sigma_{nM} = -\frac{p_n}{\zeta}\left[c_{1M} - c_{2M} + 2\,(c_{1M} + 2c_{2M})\left(\frac{x_M}{x_n}\right)^3\right], \tag{6.29}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta}\left[c_{1M} - c_{2M} - (c_{1M} + 2\,c_{2M})\left(\frac{x_M}{x_n}\right)^3\right], \tag{6.30}$$

$$\sigma_{1M} = \eta_{1M} + \frac{\eta_{3M}}{x_n^3}, \tag{6.31}$$

$$w_M = \kappa_{1M} + \frac{\kappa_{3M}}{x_n^6} + \frac{\kappa_{5M}}{x_n^3}, \tag{6.32}$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\kappa_{1M}}{3}\left(x_M^3 - x_{IN}^3\right) + \frac{\kappa_{3M}}{3}\left(\frac{1}{x_{IN}^3} - \frac{1}{x_M^3}\right) + \kappa_{5M}\ln\left(\frac{x_M}{x_{IN}}\right)\right] d\varphi\, d\nu, \tag{6.33}$$

where Θ, x_{IN}, x_M and s_{44M}, c_{iM} ($i = 1,2,3$) are given by Equations (1.15)–(1.17) and (2.13), (2.18), respectively, and ζ, η_{3M}, κ_{jM} ($j = 3,5$; Equation (6.13)) have the forms

$$\zeta = c_{1M} - c_{2M} + 2\,(c_{1M} + 2\,c_{2M})\left(\frac{x_M}{x_{IN}}\right)^3,$$

$$\eta_{3M} = \frac{p_n x_M^3(\gamma_{2M} - 2\gamma_{1M})}{\zeta} + \frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{\zeta}\right) + \gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^3}{\zeta}\right)\right],$$

$$\kappa_{3M} = 3\,(c_{1M} + 2\,c_{2M})\left(\frac{p_n x_M^3}{\zeta}\right)^2 + \frac{1}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{\zeta}\right)\right]^2 + \frac{\Theta^2}{s_{44M}}\left[\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^3}{\zeta}\right)\right]^2,$$

$$\kappa_{5M} = -\frac{2}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{\zeta}\right) + \Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^3}{\zeta}\right)\right]. \tag{6.34}$$

The coefficients η_{1M}, κ_{1M} are given by Equation (6.23), where ζ in Equation (6.23) is given by Equation (6.34). The normal stress p_n is given by Equation (2.33). With regard to Equation (6.26), the coefficient ρ_M in Equation (2.33) is derived as

$$\rho_M = \frac{1}{\zeta}\left[1 - \left(\frac{x_M}{x_{IN}}\right)^{-3}\right]. \tag{6.35}$$

Conditions $C_{2M} \neq 0$, $C_{3M} \neq 0$, $C_{1M} = 0$. With regard to Equations (2.30), (2.31), (6.5)–(6.12), (2.21), (2.26), (2.27), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta}\left[c_{3M}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} - 2\left(\frac{x_M}{x_n}\right)^3\right], \tag{6.36}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta}\left[\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} - \left(\frac{x_M}{x_n}\right)^3\right], \tag{6.37}$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = -\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)x_n^{c_{3M}-1} - \frac{1}{x_n^3}\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{\zeta}\right)\right], \tag{6.38}$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = -\Theta\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)x_n^{c_{3M}-1} - \frac{1}{x_n^3}\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^3}{\zeta}\right)\right], \tag{6.39}$$

$$\sigma_{nM} = -\frac{p_n}{\zeta}\left\{\left[c_{3M}\left(c_{1M}+c_{2M}\right) - 2c_{2M}\right]\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} - 2\left(c_{1M}+2c_{2M}\right)\left(\frac{x_M}{x_n}\right)^3\right\}, \tag{6.40}$$

$$\sigma_{\varphi M} = \sigma'_{\theta M} = -\frac{p_n}{\zeta}\left[\left(c_{1M} - c_{2M}c_{3M}\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} + \left(c_{1M}+2c_{2M}\right)\left(\frac{x_M}{x_n}\right)^3\right], \tag{6.41}$$

$$\sigma_{1M} = \eta_{2M}x_n^{c_{3M}-1} + \frac{\eta_{3M}}{x_n^3}, \tag{6.42}$$

$$w_M = \kappa_{2M}x_n^{2(c_{3M}-1)} + \frac{\kappa_{3M}}{x_n^6} + \kappa_{6M}x_n^{c_{3M}-4}, \tag{6.43}$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\kappa_{2M}}{2c_{3M}+1}\left(x_M^{2c_{3M}+1} - x_{IN}^{2c_{3M}+1}\right) + \frac{\kappa_{3M}}{3}\left(\frac{1}{x_{IN}^3} - \frac{1}{x_M^3}\right) + \frac{\kappa_{6M}}{c_{3M}-1}\left(x_M^{c_{3M}-1} - x_{IN}^{c_{3M}-1}\right)\right]d\varphi\, d\nu, \tag{6.44}$$

where Θ, x_{IN}, x_M and s_{44M}, c_{iM} $(i = 1,2,3)$ are given by Equations (1.15)–(1.17) and (2.13), (2.18), respectively, and ζ has the form

$$\zeta = \left\{ \left[c_{3M}(c_{1M}+c_{2M}) - 2c_{2M}\right] \left(\frac{x_{IN}}{x_M}\right)^{c_{3M}+2} + 2(c_{1M}+2c_{2M}) \right\} \left(\frac{x_M}{x_{IN}}\right)^3,$$

$$\kappa_{6M} = -\frac{x_M^3 \left[2c_{2M}(1-c_{3M}) - c_{1M}\right]}{x_M^{c_{3M}-1}} \left(\frac{p_n}{\zeta}\right)^2 - \frac{2}{s_{44M}} \frac{\partial}{\partial\varphi} \left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right) \frac{\partial}{\partial\varphi} \left(\frac{p_n x_M^3}{\zeta}\right)$$
$$- \frac{2\Theta^2}{s_{44M}} \frac{\partial}{\partial\nu} \left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right) \frac{\partial}{\partial\nu} \left(\frac{p_n x_M^3}{\zeta}\right). \qquad (6.45)$$

The coefficients η_{2M}, κ_{2M} and η_{3M}, κ_{3M} are given by Equations (6.23) and (6.34), respectively, where ζ in Equations (6.23), (6.34) is given by Equation (6.45). The normal stress p_n is given by Equation (2.33). With regard to Equation (6.37), the coefficient ρ_M in Equation (2.33) is derived as

$$\rho_M = \frac{1}{\zeta} \left[\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1} - \left(\frac{x_M}{x_{IN}}\right)^3 \right]. \qquad (6.46)$$

Conditions $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} \neq 0$. With regard to Equations (2.30)–(2.32), (6.5)–(6.12), (2.21), (2.26), (2.27), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta} \left\{ 1 - \frac{1}{c_{3M}+2} \left[3c_{3M} \left(\frac{x_n}{x_M}\right)^{c_{3M}-1} - 2(c_{3M}-1) \left(\frac{x_M}{x_n}\right)^3 \right] \right\}, \qquad (6.47)$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta} \left\{ 1 - \frac{1}{c_{3M}+2} \left[3 \left(\frac{x_n}{x_M}\right)^{c_{3M}-1} + (c_{3M}-1) \left(\frac{x_M}{x_n}\right)^3 \right] \right\}, \qquad (6.48)$$

$$\varepsilon_{n\varphi M} = s_{44M}\sigma_{n\varphi M} = -\left\{ \frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right) - \frac{1}{c_{3M}+2} \left[3\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right) x_n^{c_{3M}-1} + \frac{c_{3M}-1}{x_n^3} \frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{\zeta}\right) \right] \right\}, \qquad (6.49)$$

$$\varepsilon_{n\theta M} = s_{44M}\sigma_{n\theta M} = -\Theta \left\{ \frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right) \right.$$

$$
-\frac{1}{c_{3M}+2}\left[3\,\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\,x_M^{c_{3M}-1}}\right)x_n^{c_{3M}-1}+\frac{c_{3M}-1}{x_n^3}\,\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^3}{\zeta}\right)\right]\Bigg\}, \tag{6.50}
$$

$$
\sigma_{nM}=-\frac{p_n}{\zeta}\left\{c_{1M}-c_{2M}-\frac{3}{c_{3M}+2}\left[c_{3M}\left(c_{1M}+c_{2M}\right)-2c_{2M}\right]\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right.
\left.+\frac{2\left(c_{1M}+2c_{2M}\right)}{c_{3M}+2}\left(\frac{x_M}{x_n}\right)^3\right\}, \tag{6.51}
$$

$$
\sigma_{\varphi M}=\sigma_{\theta M}=-\frac{p_n}{\zeta}\left\{c_{1M}-c_{2M}-\frac{3\left(c_{1M}-c_{2M}c_{3M}\right)}{c_{3M}+2}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right.
\left.-\frac{c_{1M}+2c_{2M}}{c_{3M}+2}\left(\frac{x_M}{x_n}\right)^3\right\}, \tag{6.52}
$$

$$
\sigma_{1M}=\eta_{1M}+\eta_{2M}x_n^{c_{3M}-1}+\frac{\eta_{3M}}{x_n^3}, \tag{6.53}
$$

$$
w_M=\kappa_{1M}+\kappa_{2M}x_n^{2(c_{3M}-1)}+\frac{\kappa_{3M}}{x_n^6}+\kappa_{4M}x_n^{c_{3M}-1}+\frac{\kappa_{5M}}{x_n^3}+\kappa_{6M}x_n^{c_{3M}-4}, \tag{6.54}
$$

$$
W_M=4\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\kappa_{1M}}{3}\left(x_M^3-x_{IN}^3\right)+\frac{\kappa_{2M}}{2\,c_{3M}+1}\left(x_M^{2\,c_{3M}+1}-x_{IN}^{2\,c_{3M}+1}\right)\right.
+\frac{\kappa_{3M}}{3}\left(\frac{1}{x_{IN}^3}-\frac{1}{x_M^3}\right)+\frac{\kappa_{4M}}{c_{3M}+2}\left(x_M^{c_{3M}+2}-x_{IN}^{c_{3M}+2}\right)
\left.+\kappa_{5M}\ln\left(\frac{x_M}{x_{IN}}\right)+\frac{\kappa_{6M}}{c_{3M}-1}\left(x_M^{c_{3M}-1}-x_{IN}^{c_{3M}-1}\right)\right]d\varphi\,d\nu, \tag{6.55}
$$

where Θ, x_{IN}, x_M and s_{44M}, c_{iM} ($i=1,2,3$) are given by Equations (1.15)–(1.17) and (2.13), (2.18), respectively, and ζ, κ_{jM} ($j=2\ldots,6$; Equation (6.13)) have the forms

$$
\zeta=c_{1M}-c_{2M}+\frac{1}{c_{3M}+2}\left(\frac{x_M}{x_{IN}}\right)^3\left\{2\left(c_{3M}-1\right)\left(c_{1M}+2c_{Mm}\right)\right.
\left.-3\left[c_{3M}\left(c_{1M}+c_{2M}\right)-2\,c_{2M}\right]\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}+2}\right\},
$$

$$\eta_2 = \frac{3}{c_{3M}+2}\left\{\frac{p_n(\gamma_1 c_3+\gamma_2)}{\zeta\, x_M^{c_{3M}-1}} + \frac{1}{s_{44}}\left[\gamma_3 \frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right) + \gamma_4 \frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\right]\right\},$$

$$\eta_3 = \frac{c_{3M}-1}{c_{3M}+2}\left\{\frac{p_n x_M^3(\gamma_2-2\gamma_1)}{\zeta} + \frac{1}{s_{44}}\left[\gamma_3 \frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{\zeta}\right) + \gamma_4 \frac{\partial}{\partial\nu}\left(\frac{p_n x_M^3}{\zeta}\right)\right]\right\},$$

$$\kappa_{2M} = \left(\frac{3}{c_{3M}+2}\right)^2 \left(\left[\frac{(c_{1M}+c_{2M})\,c_{3M}^2}{2} + c_{1M} - 2\,c_{2M}\,c_{3M}\right]\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)^2\right.$$
$$\left. + \frac{1}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\right]^2\right\}\right),$$

$$\kappa_{3M} = \left(\frac{c_{3M}-1}{c_{3M}+2}\right)^2 \left(3\,(c_{1M}+2\,c_{2M})\left(\frac{p_n x_M^3}{\zeta}\right)^2\right.$$
$$\left. + \frac{1}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^3}{\zeta}\right)\right]^2\right\}\right),$$

$$\kappa_{4M} = \frac{3\,(c_{2M}-c_{1M})}{x_M^{c_{3M}-1}}\left(\frac{p_n}{\zeta}\right)^2$$
$$- \frac{6}{s_{44M}\,(c_{3M}+2)}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right) + \Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\right],$$

$$\kappa_{5M} = \frac{2\,(1-c_{3M})}{s_{44M}\,(c_{3M}+2)}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{\zeta}\right) + \Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^3}{\zeta}\right)\right],$$

$$\kappa_{6M} = \frac{3\,(c_{3M}-1)\,[2\,c_{2M}\,(1-c_{3M})-c_{1M}]}{x_M^{c_{3M}-4}}\left[\frac{p_n}{\zeta\,(c_{3M}+2)}\right]^2$$
$$+ \frac{6\,(c_{3M}-1)}{s_{44M}\,(c_{3M}+2)^2}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{\zeta}\right)$$
$$+ \frac{6\,\Theta^2\,(c_{3M}-1)}{s_{44M}\,(c_{3M}+2)^2}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^3}{\zeta}\right). \tag{6.56}$$

The coefficients η_{1M}, κ_{1M} are given by Equation (6.23), where ζ in Equation (6.23) is given by Equation (6.56). The normal stress p_n is given by Equation (2.33). With regard to Equation (6.48), the coefficient ρ_M in Equation (2.33) is derived as

$$\rho_M = \frac{1}{\zeta}\left\{1 - \frac{1}{c_{3M}+2}\left[3\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1} + (c_{3M}-1)\left(\frac{x_M}{x_{IN}}\right)^3\right]\right\}. \tag{6.57}$$

6.3 Inclusion

In case of the ellipsoidal inclusion, we get $C_{2IN} = C_{3IN} = 0$, otherwise we get $(u_{nIN})_{x_n \to 0} \longrightarrow \pm\infty$, $(\varepsilon_{IN})_{x_n \longrightarrow 0} \longrightarrow \pm\infty$, $(\sigma_{IN})_{r \to 0} \longrightarrow \pm\infty$ due to $c_3 < 0$ (see Equations (2.18), (6.4)–(6.10)). With regard to Equations (2.28), (2.29), (6.4)–(6.12), (2.21), (2.26), (2.27), we get [1]–[22]

$$\varepsilon_{nIN} = \varepsilon_{\varphi IN} = \varepsilon_{\theta IN} = -p_n \rho_{IN}, \tag{6.58}$$

$$\varepsilon_{n\varphi IN} = s_{44IN} \sigma_{n\varphi IN} = -\rho_{IN} \frac{\partial p_n}{\partial \varphi}, \tag{6.59}$$

$$\varepsilon_{n\theta IN} = s_{44IN} \sigma_{n\theta IN} = -\Theta \, \rho_{IN} \frac{\partial p_n}{\partial \nu}, \tag{6.60}$$

$$\sigma_{nIN} = \sigma_{\varphi IN} = \sigma_{\theta IN} = -p_n, \tag{6.61}$$

$$\sigma_{1IN} = -\rho_{IN} \left[p_n (\gamma_1 + \gamma_2) + \frac{1}{s_{44}} \left(\gamma_3 \frac{\partial p_n}{\partial \varphi} + \gamma_4 \frac{\partial p_n}{\partial \nu} \right) \right], \tag{6.62}$$

$$w_{IN} = \rho_{IN}^2 \left\{ \frac{3 p_n^2}{2 \rho_{IN}} + \frac{2}{s_{44IN}} \left[\left(\frac{\partial p_n}{\partial \varphi} \right)^2 + \left(\frac{\partial p_n}{\partial \nu} \right)^2 \right] \right\}, \tag{6.63}$$

$$W_{IN} = \frac{4 \rho_{IN}^2}{3} \int_0^{\pi/2} \int_0^{\pi/2} x_{IN}^3 \left\{ \frac{3 p_n^2}{2 \rho_{IN}} + \frac{2}{s_{44IN}} \left[\left(\frac{\partial p_n}{\partial \varphi} \right)^2 + \left(\frac{\partial p_n}{\partial \nu} \right)^2 \right] \right\} d\varphi \, d\nu, \tag{6.64}$$

where Θ, s_{44IN} are given by Equations (1.15), (2.13), respectively. The normal stress p_n is given by Equation (2.33). With regard to Equation (6.58), the coefficient ρ_{IN} in Equation (2.33) is derived as

$$\rho_{IN} = \frac{1 - 2\mu_{IN}}{E_{IN}}. \tag{6.65}$$

Chapter 7

Mathematical Model 5

7.1 Mathematical Procedure

Let the mathematical procedures ∂Eq. (2.24) $/\partial r$, Eq. (6.2) $/r$ be performed, and then we get

$$x_n \frac{\partial^2 U_n}{\partial x_n^2} + (1 - c_3) \frac{\partial U_n}{\partial x_n} = 0, \tag{7.1}$$

$$\frac{\partial U_n}{\partial x_n} = -s_{44} (c_1 + c_2) \left(x_n^2 \frac{\partial^3 u_n}{\partial x_n^3} + 4 x_n \frac{\partial^2 u_n}{\partial x_n^2} \right), \tag{7.2}$$

where s_{44} and c_1, c_2, $c_3 < 0$ are given by Equations (2.13) and (2.18), respectively. Let the mathematical procedure ∂Eq. (7.2) $/\partial r$ be performed, and then we get

$$\frac{\partial^2 U_n}{\partial x_n^2} = -s_{44} (c_1 + c_2) \left(x_n^2 \frac{\partial^4 u_n}{\partial x_n^4} + 6 x_n \frac{\partial^3 u_n}{\partial x_n^3} + 4 \frac{\partial^2 u_n}{\partial x_n^2} \right). \tag{7.3}$$

Let Equations (6.2), (6.3) be substituted to (7.1), and then we get

$$x_n^2 \frac{\partial^4 u_n}{\partial x_n^4} + (7 - c_3) x_n \frac{\partial^3 u_n}{\partial x_n^3} + 4 (2 - c_3) \frac{\partial^2 u_n}{\partial x_n^2} = 0. \tag{7.4}$$

Let u_n be assumed in the form $u_n = x_n^{\lambda}$, then we get

$$u_n = C_1 x_n + C_2 x_n^{c_3} + \frac{C_3}{x_n^2} + C_4, \tag{7.5}$$

where $C_1 \ldots, C_4$ are integration constants, which are determined by the boundary conditions in Section 2.3. With regard to Equations (2.1)–(2.4), (2.14)–(2.17), (2.26), (7.6), we get

$$\varepsilon_n = C_1 + C_2 c_3 x_n^{c_3 - 1} - \frac{2 C_3}{x_n^3}, \tag{7.6}$$

$$\varepsilon_\varphi = \varepsilon_\theta = C_1 + C_2 x_n^{c_3 - 1} + \frac{C_3}{x_n^3} + \frac{C_4}{x_n}, \tag{7.7}$$

$$\varepsilon_{n\varphi} = s_{44} \sigma_{n\varphi} = \frac{\partial C_1}{\partial \varphi} + x_n^{c_3 - 1} \frac{\partial C_2}{\partial \varphi} + \frac{1}{x_n^3} \frac{\partial C_3}{\partial \varphi} + \frac{1}{x_n} \frac{\partial C_4}{\partial \varphi}, \tag{7.8}$$

$$\varepsilon_{n\theta} = s_{44}\,\sigma_{n\theta} = \Theta\left[\frac{\partial C_1}{\partial \nu} + x_n^{c_3-1}\,\frac{\partial C_2}{\partial \nu} + \frac{1}{x_n^3}\,\frac{\partial C_3}{\partial \nu} + \frac{1}{x_n}\,\frac{\partial C_4}{\partial \nu}\right], \tag{7.9}$$

$$\sigma_n = C_1\,(c_1 - c_2) + C_2\,[(c_1 + c_2)\,c_3 - 2\,c_2]\,x_n^{c_3-1} - \frac{2\,C_3\,(c_1 + 2\,c_2)}{x_n^3} - \frac{2\,c_2\,C_4}{x_n}, \tag{7.10}$$

$$\sigma_\varphi = \sigma_\theta = C_1\,(c_1 - c_2) + C_2\,(c_1 - c_2\,c_3)\,x_n^{c_3-1} + \frac{C_3\,(c_1 + 2\,c_2)}{x_n^3} + \frac{c_1\,C_4}{x_n}, \tag{7.11}$$

$$\sigma_1 = \eta_1 + \eta_2\,x_n^{c_3-1} + \frac{\eta_3}{x_n^3} + \frac{\eta_4}{x_n}, \tag{7.12}$$

$$w = \kappa_1 + \kappa_2\,x_n^{2(c_3-1)} + \frac{\kappa_3}{x_n^6} + \frac{\kappa_4}{x_n^2} + (\kappa_5 + \kappa_9)\,x_n^{c_3-1} + \frac{\kappa_6}{x_n^3} + \kappa_7\,x_n^{c_3-4} + \frac{\kappa_8}{x_n} + \frac{\kappa_{10}}{x_n^4}, \tag{7.13}$$

where Θ and η_i ($i = 1,2,3$) are given by Equations (1.15) and (6.13), respectively, and η_4, κ_j ($j = 4,5,6$) are derived as

$$\begin{aligned}
\eta_4 &= C_4\,\gamma_2 + \frac{1}{s_{44}}\left(\gamma_3\,\frac{\partial C_4}{\partial \varphi} + \gamma_4\,\frac{\partial C_4}{\partial \nu}\right),\\
\kappa_4 &= c_1\,C_4^2 + \frac{1}{s_{44}}\left[\left(\frac{\partial C_4}{\partial \varphi}\right)^2 + \Theta^2\left(\frac{\partial C_4}{\partial \nu}\right)^2\right],\\
\kappa_5 &= (c_1 - c_2)\,(2 + c_3)\,C_1\,C_2 + \frac{2}{s_{44}}\left(\frac{\partial C_1}{\partial \varphi}\,\frac{\partial C_2}{\partial \varphi} + \Theta^2\,\frac{\partial C_1}{\partial \nu}\,\frac{\partial C_2}{\partial \nu}\right),\\
\kappa_6 &= \frac{2}{s_{44}}\left(\frac{\partial C_1}{\partial \varphi}\,\frac{\partial C_3}{\partial \varphi} + \Theta^2\,\frac{\partial C_1}{\partial \nu}\,\frac{\partial C_3}{\partial \nu}\right),\\
\kappa_7 &= [2\,c_2\,(1 - c_3) - c_1]\,C_2\,C_3 + \frac{2}{s_{44}}\left(\frac{\partial C_2}{\partial \varphi}\,\frac{\partial C_3}{\partial \varphi} + \Theta^2\,\frac{\partial C_2}{\partial \nu}\,\frac{\partial C_3}{\partial \nu}\right),\\
\kappa_8 &= (c_1 - c_2)\,C_1\,C_4 + \frac{1}{s_{44}}\left(\frac{\partial C_1}{\partial \varphi}\,\frac{\partial C_4}{\partial \varphi} + \Theta^2\,\frac{\partial C_1}{\partial \nu}\,\frac{\partial C_4}{\partial \nu}\right),\\
\kappa_9 &= (c_1 - c_2\,c_3)\,C_2\,C_4 + \frac{1}{s_{44}}\left(\frac{\partial C_2}{\partial \varphi}\,\frac{\partial C_4}{\partial \varphi} + \Theta^2\,\frac{\partial C_2}{\partial \nu}\,\frac{\partial C_4}{\partial \nu}\right),\\
\kappa_{10} &= (c_1 + 2\,c_2)\,C_3\,C_4 + \frac{1}{s_{44}}\left(\frac{\partial C_3}{\partial \varphi}\,\frac{\partial C_4}{\partial \varphi} + \Theta^2\,\frac{\partial C_3}{\partial \nu}\,\frac{\partial C_4}{\partial \nu}\right).
\end{aligned} \tag{7.14}$$

The coefficient κ_i ($i = 1,2,3$) is given by Equation (6.13). In case of the ellipsoidal inclusion, we get $C_{2IN} = C_{3IN} = C_{4IN} = 0$, otherwise we get $(u_{nIN})_{x_n \to 0} \longrightarrow \pm\,\infty$, $(\varepsilon_{IN})_{x_n \longrightarrow 0} \longrightarrow \pm\,\infty$, $(\sigma_{IN})_{r \to 0} \longrightarrow \pm\infty$ due to $c_3 < 0$ (see Equations (2.18), (6.4)–(6.10)). In case of $C_{1IN} \neq 0$ (see Equations (6.4), (7.5)), the mathematical solutions for the ellipsoidal inclusion is presented in Section 6.3.

7.2 Matrix

The integration constants $C_{1M}, C_{2M}, C_{3M}, C_{4M}$ for the matrix (see Equation (4.4)) are determined by the boundary conditions (2.30), (2.31) or (2.30)–(2.32). The boundary conditions result in the following combinations of C_{1M}, C_{2M}, C_{3M}, C_{4M}, where the combinations of C_{1M}, C_{2M}, C_{3M} are presented in Section (6.2). Finally, such a combination is considered to exhibit a minimum value of the elastic energy W_C of the cubic cell (see Equation (2.27)).

Conditions $C_{1M} \neq 0$, $C_{4M} \neq 0$, $C_{2M} = C_{3M} = 0$. With regard to Equations (2.30), (2.31), (7.5)–(7.13), (2.21), (2.26), (2.27), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta}, \tag{7.15}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta}\left(1 - \frac{1}{x_n}\right), \tag{7.16}$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = -\left(1 - \frac{1}{x_n}\right)\frac{\partial}{\partial \varphi}\left(\frac{p_n}{\zeta}\right), \tag{7.17}$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = -\Theta\left(1 - \frac{1}{x_n}\right)\frac{\partial}{\partial \nu}\left(\frac{p_n}{\zeta}\right), \tag{7.18}$$

$$\sigma_{nM} = -\frac{p_n}{\zeta}\left(c_{1M} - c_{2M} + \frac{2c_{Mm}}{x_n}\right), \tag{7.19}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta}\left(c_{1M} - c_{2M} - \frac{c_{1M}}{x_n}\right), \tag{7.20}$$

$$\sigma_{1M} = \eta_{1M} + \frac{\eta_{4M}}{x_n}, \tag{7.21}$$

$$w_M = \kappa_{1M} + \frac{\kappa_4}{x_n^2} + \frac{\kappa_8}{x_n}, \tag{7.22}$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\kappa_{1M}}{3}\left(x_M^3 - x_{IN}^3\right) + \kappa_{4M}\left(x_M - x_{IN}\right) + \frac{\kappa_{8M}}{2}\left(x_M^2 - x_{IN}^2\right)\right] d\varphi\, d\nu, \tag{7.23}$$

where Θ, x_{IN}, x_M and s_{44M}, c_{iM} $(i=1,2)$ are given by Equations (1.15)–(1.17) and (2.13), (2.18), respectively, and ζ, η_{4M}, κ_{jM} $(j=4,8$; see Equation (7.14)) have the forms

$$\zeta = c_{1M} - c_{2M} + \frac{2c_{2M}x_M}{x_{IN}},$$
$$\eta_{4M} = \frac{p_n\gamma_{2M}}{\zeta} + \frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right) + \gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right],$$
$$\kappa_{4M} = c_{1M}\left(\frac{p_n}{\zeta}\right)^2 + \frac{1}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right]^2 + \left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right]^2\right\},$$
$$\kappa_{8m} = (c_{2M} - c_{1M})\left(\frac{p_n}{\zeta}\right)^2 - \frac{1}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right]^2 + \left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right]^2\right\}. \quad (7.24)$$

The coefficients η_{1M}, κ_{1M} are given by Equation (6.23), where ζ in Equation (6.23) is given by Equation (7.32). The normal stress p_n is given by Equation (2.33). With regard to Equation (7.16), the coefficient ρ_M in Equation (2.33) is derived as

$$\rho_M = \frac{1}{\zeta}\left(1 - \frac{1}{x_{IN}}\right). \quad (7.25)$$

Conditions $C_{2M} \neq 0$, $C_{4M} \neq 0$, $C_{1M} = C_{3M} = 0$. With regard to Equations (2.30), (2.31), (7.5)–(7.13), (2.21), (2.26), (2.27), we get

$$\varepsilon_{nM} = -\frac{p_n c_{3M}}{\zeta}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}, \quad (7.26)$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta}\left[\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} - \frac{1}{x_n}\right], \quad (7.27)$$

$$\varepsilon_{n\varphi M} = s_{44M}\sigma_{n\varphi M} = -\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\kappa x_M^{c_{3M}-1}}\right) - \frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right], \quad (7.28)$$

$$\varepsilon_{n\theta M} = s_{44M}\sigma_{n\theta M} = -\Theta\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\kappa x_M^{c_{3M}-1}}\right) - \frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right], \quad (7.29)$$

$$\sigma_{nM} = -\frac{p_n}{\zeta}\left\{\left[c_{3M}(c_{1M} + c_{2M}) - 2c_{2M}\right]\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} + \frac{2c_{2M}}{x_n}\right\}, \quad (7.30)$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta}\left[(c_{1M} - c_{2M}c_{3M})\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} - \frac{c_{1M}}{x_n}\right], \quad (7.31)$$

$$\sigma_{1M} = \eta_{2M} x_n^{c_{3M}-1} + \frac{\eta_{4M}}{x_n}, \tag{7.32}$$

$$w_M = \kappa_{2M} x_n^{2(c_{3M}-1)} + \frac{\kappa_{4M}}{x_n^2} + \kappa_{9M}\, x_n^{c_{3M}-1}, \tag{7.33}$$

$$W_M = 4 \int_0^{\pi/2}\int_0^{\pi/2} \left[\frac{\kappa_{2M}}{2c_{3M}+1} \left(x_M^{2c_{3M}+1} - x_{IN}^{2c_{3M}+1} \right) + \kappa_{4M} \left(x_M - x_{IN} \right) \right.$$
$$\left. + \frac{\kappa_{9M}}{c_{3M}+2} \left(x_M^{c_{3M}+2} - x_{IN}^{c_{3M}+2} \right) \right] d\varphi\, d\nu, \tag{7.34}$$

where Θ, x_{IN}, x_M and s_{44M}, c_{iM} ($i=1,2,3$) are given by Equations (1.15)–(1.17) and (2.13), (2.18), respectively, and ζ, κ_{2M} (see Equation (6.13)), κ_{9M} (see Equation (7.14)) have the forms

$$\zeta = \left[c_{3M} \left(c_{1M} + c_{2M} \right) - 2c_{2M} \right] \left(\frac{x_{IN}}{x_M} \right)^{c_{3M}-1} + \frac{2c_{2M} x_M}{x_{IN}},$$

$$\kappa_{2M} = \left[\frac{c_{3M}^2 \left(c_{1M} + c_{2M} \right)}{2} + c_{1M} - 2c_{2M} c_{3M} \right] \left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}} \right)^2$$
$$+ \frac{1}{s_{44M}} \left\{ \left[\frac{\partial}{\partial\varphi} \left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}} \right) \right]^2 + \Theta^2 \left[\frac{\partial}{\partial\nu} \left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}} \right) \right]^2 \right\},$$

$$\kappa_{9M} = -\frac{c_{1M} - c_{2M} c_{3M}}{x_M^{c_{3M}-1}} \left(\frac{p_n}{\kappa_M} \right)^2 - \frac{1}{s_{44M}} \frac{\partial}{\partial\varphi} \left(\frac{p_n}{\kappa_M x_M^{c_{3M}-1}} \right) \frac{\partial}{\partial\varphi} \left(\frac{p_n}{\kappa_M} \right)$$
$$- \frac{\Theta^2}{s_{44M}} \frac{\partial}{\partial\nu} \left(\frac{p_n}{\kappa_M x_M^{c_{3M}-1}} \right) \frac{\partial}{\partial\nu} \left(\frac{p_n}{\kappa_M} \right). \tag{7.35}$$

The coefficients η_{2M} and η_{4M}, κ_{4M} are given by Equations (6.23) and (7.24), respectively, where ζ in Equations (6.23), (7.24) is given by Equation (7.44). The normal stress p_n is given by Equation (2.33). With regard to Equation (7.27), the coefficient ρ_M in Equation (2.33) is derived as

$$\rho_M = \frac{1}{\zeta} \left[\left(\frac{x_{IN}}{x_M} \right)^{c_{3M}-1} - \frac{1}{x_{IN}} \right]. \tag{7.36}$$

Conditions $C_{3M} \neq 0$**,** $C_{4M} \neq 0$**,** $C_{1M} = C_{2M} = 0$**.** With regard to Equations (2.30), (2.31), (7.5)–(7.13), (2.21), (2.26), (2.27), we get

$$\varepsilon_{nM} = \frac{2p_n}{\zeta}\left(\frac{x_{IN}}{x_n}\right)^3, \tag{7.37}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta}\left[\left(\frac{x_{IN}}{x_n}\right)^3 - \frac{1}{x_n}\right], \tag{7.38}$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = -\left[\left(\frac{x_{IN}}{x_n}\right)^3 - \frac{1}{x_n}\right]\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right), \tag{7.39}$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = -\left[\left(\frac{x_{IN}}{x_n}\right)^3 - \frac{1}{x_n}\right]\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right), \tag{7.40}$$

$$\sigma_{nM} = \frac{2p_n}{\zeta}\left[(c_{1M} + 2c_{2M})\left(\frac{x_{IN}}{x_n}\right)^3 - \frac{c_{2M}}{x_n}\right], \tag{7.41}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta}\left[(c_{1M} - 2c_{2M})\left(\frac{x_{IN}}{x_n}\right)^3 - \frac{c_{1M}}{x_n}\right], \tag{7.42}$$

$$\sigma_{1M} = \frac{\eta_3}{x_n^3} + \frac{\eta_4}{x_n}, \tag{7.43}$$

$$w_M = \frac{\kappa_{3M}}{x_n^6} + \frac{\kappa_{4M}}{x_n^2} + \frac{\kappa_{10M}}{x_n^4}, \tag{7.44}$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\kappa_{3M}}{3}\left(\frac{1}{x_{IN}^3} - \frac{1}{x_M^3}\right) + \kappa_{4M}(x_M - x_{IN}) + \kappa_{10M}\left(\frac{1}{x_{IN}} - \frac{1}{x_M}\right)\right]d\varphi\, d\nu, \tag{7.45}$$

where Θ, x_{IN}, x_M and s_{44M}, $c_{iM}\,(i = 1,2)$ are given by Equations (1.15)–(1.17) and (2.13), (2.18), respectively, and ζ, κ_{3M} (see Equation (6.13)), κ_{10M} (see Equation (7.14)) have the forms

$$\zeta = -\left[2(c_{1M} + 2c_{2M}) + 2c_{2M}\left(\frac{x_{IN}}{x_M}\right)^2\right],$$

$$\kappa_{3M} = 3(c_{1M} + 2c_{2M})\left(\frac{p_n x_{IN}^3}{\zeta}\right)^2$$

$$+\frac{1}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n x_{IN}^3}{\zeta}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n x_{IN}^3}{\zeta}\right)\right]^2\right\},$$

$$\kappa_{10M}=-x_{IN}^3\,(c_{1M}+2\,c_{2M})\left(\frac{p_n}{\zeta}\right)^2-\frac{1}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n x_{IN}^3}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right]$$
$$-\frac{\Theta^2}{s_{44M}}\left[\frac{\partial}{\partial\nu}\left(\frac{p_n x_{IN}^3}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right]. \tag{7.46}$$

The coefficients η_{3M} and η_{4M}, κ_{4M} are given by Equations (6.34), (7.24), respectively, where ζ in Equations (6.34), (7.24) is given by Equation (7.46). The normal stress p_n is given by Equation (2.33). With regard to Equation (7.38), the coefficient ρ_M in Equation (2.33) is derived as

$$\rho_M=\frac{x_{IN}-1}{\zeta\,x_{IN}}. \tag{7.47}$$

Conditions $C_{1M}\neq 0, C_{2M}\neq 0, C_{4M}\neq 0, C_{3M}=0$. With regard to Equations (2.30)–(2.32), (7.5)–(7.13), (2.21), (2.26), (2.27), we get

$$\varepsilon_{nM}=-\frac{p_n}{\zeta}\left[1-\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right], \tag{7.48}$$

$$\varepsilon_{\varphi M}=\varepsilon_{\theta M}=-\frac{p_n}{\zeta}\left\{1-\frac{1}{c_{3M}}\left[\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}+\frac{(c_{3M}-1)\,x_M}{x_n}\right]\right\}, \tag{7.49}$$

$$\varepsilon_{n\varphi M}=s_{44M}\,\sigma_{n\varphi M}=$$
$$-\left\{\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)-\frac{1}{c_{3M}}\left[x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\,x_M^{c_{3M}-1}}\right)+\frac{c_{3M}-1}{x_n}\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M}{\zeta}\right)\right]\right\}, \tag{7.50}$$

$$\varepsilon_{n\theta M}=s_{44M}\,\sigma_{n\theta M}=$$
$$-\left\{\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)-\frac{1}{c_{3M}}\left[x_n^{c_{3M}-1}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\,x_M^{c_{3M}-1}}\right)+\frac{c_{3M}-1}{x_n}\frac{\partial}{\partial\nu}\left(\frac{p_n x_M}{\zeta}\right)\right]\right\}, \tag{7.51}$$

$$\sigma_{nM}=-\frac{p_n}{\zeta}\left(c_{1M}-c_{2M}-\frac{1}{c_{3M}}\left\{[c_{3M}\,(c_{1M}+c_{2M})-2c_{2M}]\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right.\right.$$
$$\left.\left.+\frac{2c_{2M}\,(c_{3M}-1)\,x_M}{c_{3M}\,x_n}\right\}\right), \tag{7.52}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta}\left\{c_{1M} - c_{2M} - \frac{1}{c_{3M}}\left[(c_{1M} - c_{2M}c_{3M})\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} + \frac{c_{1M}(c_{1M}-1)\,x_M}{x_n}\right]\right\}, \quad (7.53)$$

$$\sigma_{1M} = \eta_{1M} + \eta_{2M}x_n^{c_{3M}-1} + \frac{\eta_{4M}}{x_n}, \quad (7.54)$$

$$w_M = \kappa_{1M} + \kappa_{2M}x_n^{2(c_{3M}-1)} + \frac{\kappa_{4M}}{x_n^2} + (\kappa_{5M} + \kappa_{9M})x_n^{c_{3M}-1} + \frac{\kappa_{8M}}{x_n}, \quad (7.55)$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\kappa_{1M}}{3}\left(x_M^3 - x_{IN}^3\right) + \frac{\kappa_{2M}}{2c_{3M}+1}\left(x_M^{2c_{3M}+1} - x_{IN}^{2c_{3M}+1}\right) + \kappa_{4M}(x_M - x_{IN}) + \frac{\kappa_{5M}+\kappa_{9M}}{c_{3M}+2}\left(x_M^{c_{3M}+2} - x_{IN}^{c_{3M}+2}\right) + \frac{\kappa_{8M}}{2}\left(x_M^2 - x_{IN}^2\right)\right]d\varphi\, d\nu, \quad (7.56)$$

where Θ, x_{IN}, x_M and s_{44M}, c_{iM} $(i=1,2,3)$ are given by Equations (1.15)–(1.17) and (2.13), (2.18), respectively, and ζ, κ_{iM} $(i=1,2$; see Equation (6.13)), κ_{jM} $(j=4,5,8,9$; see Equation (7.14)) have the forms

$$\zeta = (c_{1M} - c_{2M}) - \frac{[c_{3M}(c_{1M}+c_{2M}) - 2c_{2M}]}{c_{3M}}\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1} + \frac{2c_{2M}(c_{3M}-1)\,x_M}{c_{3M}x_{IN}},$$

$$\kappa_{4M} = c_{1M}\left[\frac{p_n x_M(c_{3M}-1)}{\zeta\, c_{3M}}\right]^2 + \frac{1}{s_{44M}}\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n x_M(c_{3M}-1)}{\zeta\, c_{3M}}\right]\right\}^2 + \frac{\Theta^2}{s_{44M}}\left\{\frac{\partial}{\partial\nu}\left[\frac{p_n x_M(c_{3M}-1)}{\zeta\, c_{3M}}\right]\right\}^2,$$

$$\kappa_{5M} = -\frac{(c_{1M}-c_{2M})(2+c_{3M})}{c_{3M}\,x_M^{c_{3M}-1}}\left(\frac{p_n}{\zeta}\right)^2 - \frac{2}{s_{44M}}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, c_{3M}\, x_M^{c_{3M}-1}}\right) - \frac{2\Theta^2}{s_{44M}}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, c_{3M}\, x_M^{c_{3M}-1}}\right),$$

$$\kappa_{8M} = -\,(c_{1M}-c_{2M})\left(\frac{p_n}{\zeta}\right)\left[\frac{p_n x_M(c_{3M}-1)}{\zeta\, c_{3M}}\right] - \frac{1}{s_{44M}}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n x_M(c_{3M}-1)}{\zeta\, c_{3M}}\right]$$

$$- \frac{\Theta^2}{s_{44M}} \frac{\partial}{\partial \nu} \left(\frac{p_n}{\zeta} \right) \frac{\partial}{\partial \nu} \left[\frac{p_n x_M (c_{3M} - 1)}{\zeta \, c_{3M}} \right],$$

$$\kappa_{9M} = \frac{(c_{1M} - c_{2M} c_{3M})(c_{3M} - 1)}{x_M^{c_{3M}-2}} \left(\frac{p_n}{\zeta \, c_{3M}} \right)^2 + \frac{(c_{3M} - 1)}{s_{44M} \, c_{3M}^2} \frac{\partial}{\partial \varphi} \left(\frac{p_n}{\zeta \, x_M^{c_{3M}-1}} \right) \frac{\partial}{\partial \varphi} \left(\frac{p_n x_M}{\zeta} \right) + \frac{\Theta^2 (c_{3M} - 1)}{s_{44M} \, c_{3M}^2} \frac{\partial}{\partial \nu} \left(\frac{p_n}{\zeta \, x_M^{c_{3M}-1}} \right) \frac{\partial}{\partial \nu} \left(\frac{p_n x_M}{\zeta} \right). \tag{7.57}$$

The coefficients η_{iM}, κ_{iM} ($i = 1,2$) and η_4 are given by Equations (6.23) and (7.24), respectively, where ζ in Equations (6.23) and (7.24) is given by Equation (7.57). The normal stress p_n is given by Equation (2.33). With regard to Equation (7.49), the coefficient ρ_M in Equation (2.33) is derived as

$$\rho_M = \frac{1}{\zeta} \left\{ 1 - \frac{1}{c_{3M}} \left[\left(\frac{x_{IN}}{x_M} \right)^{c_{3M}-1} + \frac{(c_{3M} - 1) x_M}{x_{IN}} \right] \right\}. \tag{7.58}$$

Conditions $C_{1m} \neq 0, C_{3m} \neq 0, C_{4m} \neq 0, C_{2m} = 0$. With regard to Equations (2.30)–(2.32), (7.5)–(7.13), (2.21), (2.26), (2.27), we get

$$\varepsilon_{nM} = - \frac{p_n}{\zeta} \left[1 - \frac{3}{2} \left(\frac{x_M}{x_n} \right)^3 \right], \tag{7.59}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = - \frac{p_n}{\zeta} \left[1 + \frac{1}{2} \left(\frac{x_M}{x_n} \right)^3 - \frac{3 x_M}{2 x_n} \right], \tag{7.60}$$

$$\varepsilon_{n\varphi M} = s_{44M} \sigma_{n\varphi M} = - \left[\frac{\partial}{\partial \varphi} \left(\frac{p_n}{\zeta} \right) + \frac{1}{2 x_n^3} \frac{\partial}{\partial \varphi} \left(\frac{p_n x_M^3}{\zeta} \right) - \frac{3}{2 x_n} \frac{\partial}{\partial \varphi} \left(\frac{p_n x_M}{\zeta} \right) \right], \tag{7.61}$$

$$\varepsilon_{n\theta M} = s_{44M} \sigma_{n\theta M} = - \left[\frac{\partial}{\partial \nu} \left(\frac{p_n}{\zeta} \right) + \frac{1}{2 x_n^3} \frac{\partial}{\partial \nu} \left(\frac{p_n x_M^3}{\zeta} \right) - \frac{3}{2 x_n} \frac{\partial}{\partial \nu} \left(\frac{p_n x_M}{\zeta} \right) \right], \tag{7.62}$$

$$\sigma_{nM} = - \frac{p_n}{\zeta} \left[c_{1M} - c_{2M} - (c_{1M} + 2 c_{2M}) \left(\frac{x_M}{x_n} \right)^3 + \frac{3 c_{2M} x_M}{x_n} \right], \tag{7.63}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta}\left[c_{1M} - c_{2M} + \frac{c_{1M} + 2c_{2M}}{2}\left(\frac{x_M}{x_n}\right)^3 - \frac{3c_{1M}x_M}{2x_n}\right], \quad (7.64)$$

$$\sigma_{1M} = \eta_{1M} + \frac{\eta_{3M}}{x_n^3} + \frac{\eta_{4M}}{x_n}, \quad (7.65)$$

$$w_M = \kappa_{1M} + \frac{\kappa_{3M}}{x_n^6} + \frac{\kappa_{4M}}{x_n^2} + \frac{\kappa_{6M}}{x_n^3} + \frac{\kappa_{8M}}{x_n} + \frac{\kappa_{10M}}{x_n^4}, \quad (7.66)$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\kappa_{1M}}{3}\left(x_M^3 - x_{IN}^3\right) + \frac{\kappa_{3M}}{3}\left(\frac{1}{x_{IN}^3} - \frac{1}{x_M^3}\right) + \kappa_{4M}\left(x_M - x_{IN}\right)\right.$$
$$\left. + \kappa_{6M}\ln\left(\frac{x_M}{x_{IN}}\right) + \frac{\kappa_{8M}}{2}\left(x_M^2 - x_{IN}^2\right) + \kappa_{10M}\left(\frac{1}{x_{IN}} - \frac{1}{x_M}\right)\right] d\varphi\, d\nu, \quad (7.67)$$

where Θ, x_{IN}, x_M and s_{44M}, c_{iM} ($i=1,2$) are given by Equations (1.15)–(1.17) and (2.13), (2.18), respectively, and ζ, κ_{3M} (see Equation (6.13)), κ_{iM} ($i=4,6,8,10$; see Equation (7.14)) have the forms

$$\zeta = (c_{1M} - c_{2M}) - (c_{1M} + 2c_{2M})\left(\frac{x_M}{x_{IN}}\right)^3 + \frac{3c_{2M}x_M}{x_{IN}},$$
$$\kappa_{3M} = 3\left(c_{1M} + 2c_{2M}\right)\left(\frac{p_n x_M^3}{2\zeta}\right)^2 + \frac{1}{s_{44}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{2\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^3}{2\zeta}\right)\right]^2\right\},$$
$$\kappa_{4M} = c_{1M}\left(\frac{3p_n x_M}{2\zeta}\right)^2 + \frac{1}{s_{44}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{3p_n x_M}{2\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{3p_n x_M}{2\zeta}\right)\right]^2\right\},$$
$$\kappa_{6M} = \frac{2}{s_{44}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{2\zeta}\right) + \Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^3}{2\zeta}\right)\right],$$
$$\kappa_{8M} = -\frac{3x_M\left(c_{1M} - c_{2M}\right)}{2}\left(\frac{p_n}{\zeta}\right)^2$$
$$- \frac{3}{2s_{44}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M}{\zeta}\right) + \Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n x_M}{\zeta}\right)\right],$$
$$\kappa_{10M} = -\frac{3x_M^4\left(c_{1M} + 2c_{2M}\right)}{4}\left(\frac{p_n}{\zeta}\right)^2$$
$$- \frac{3}{4s_{44}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M}{\zeta}\right) + \Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^3}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n x_M}{\zeta}\right)\right]. \quad (7.68)$$

The coefficients η_{1M}, η_{3M}, κ_{1M} and η_{4M} are given by Equations (6.23) and (7.24), respectively, where ζ in Equations (6.23) and (7.24) is given by Equation (7.78). The normal stress p_n is given by Equation (2.33). With regard to Equation (7.60), the coefficient ρ_M in Equation (2.33) is derived as

$$\rho_M = \frac{1}{\zeta}\left[1 + \frac{1}{2}\left(\frac{x_M}{x_{IN}}\right)^3 - \frac{3x_M}{2x_{IN}}\right]. \tag{7.69}$$

Conditions $C_{2m} \neq 0, C_{3m} \neq 0, C_{4m} \neq 0, C_{1m} = 0.$ With regard to Equations (2.30)–(2.32), (7.5)–(7.13), (2.21), (2.26), (2.27), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta}\left[c_{3M}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} - c_{3M}\left(\frac{x_M}{x_n}\right)^3\right], \tag{7.70}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta}\left[\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} + \frac{c_{3M}}{2}\left(\frac{x_M}{x_n}\right)^3 - \frac{(c_{3M}+2)\,x_M}{2x_n}\right], \tag{7.71}$$

$$\begin{aligned}\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = &-\left[x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right) + \frac{c_{3M}}{2x_n^3}\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{\zeta}\right)\right.\\ &\left.+\frac{(c_{3M}+2)}{2x_n}\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M}{\zeta}\right)\right],\end{aligned} \tag{7.72}$$

$$\begin{aligned}\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = &-\left[x_n^{c_{3M}-1}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right) + \frac{c_{3M}}{2x_n^3}\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^3}{\zeta}\right)\right.\\ &\left.+\frac{(c_{3M}+2)}{2x_n}\frac{\partial}{\partial\nu}\left(\frac{p_n x_M}{\zeta}\right)\right],\end{aligned} \tag{7.73}$$

$$\begin{aligned}\sigma_{nM} = -\frac{p_n}{\zeta}&\left\{\left[c_{3M}\left(c_{1M}+c_{2M}\right)-2c_{2M}\right]\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right.\\ &\left.-\left(c_{3M}c_{1M}+2c_{2M}\right)\left(\frac{x_M}{x_n}\right)^3 + \frac{c_{2M}\left(c_{3M}+2\right)x_M}{x_n}\right\},\end{aligned} \tag{7.74}$$

$$\begin{aligned}\sigma_{\varphi M} = \sigma_{\theta M} = \frac{p_n}{\zeta}&\left[\left(c_{1M}-c_{2M}c_{3M}\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} + \frac{c_{3M}\left(c_{1M}+2c_{2M}\right)}{2}\left(\frac{x_M}{x_n}\right)^3\right.\\ &\left.-\frac{c_{1M}\left(c_{3M}+2\right)x_M}{2x_n}\right],\end{aligned} \tag{7.75}$$

$$\sigma_{1M} = \eta_{2M} x_n^{c_{3M}-1} + \frac{\eta_{3M}}{x_n^3} + \frac{\eta_{4M}}{x_n}, \tag{7.76}$$

$$w_M = \kappa_{2M} x_n^{2(c_{3M}-1)} + \frac{\kappa_{3M}}{x_n^6} + \frac{\kappa_{4M}}{x_n^2} + \kappa_{7M} x_n^{c_{3M}-4} + \kappa_{9M} x_n^{c_{3M}-1} + \frac{\kappa_{10M}}{x_n^4}, \tag{7.77}$$

$$\begin{aligned} W_M = 4 \int_0^{\pi/2} \int_0^{\pi/2} \Bigg[& \frac{\kappa_{2M}}{2c_{3M}+1} \left(x_M^{2c_{3M}+1} - x_{IN}^{2c_{3M}+1} \right) + \frac{\kappa_{3M}}{3} \left(\frac{1}{x_{IN}^3} - \frac{1}{x_M^3} \right) \\ & + \kappa_{4M} (x_M - x_{IN}) + \frac{\kappa_{7M}}{c_{3M}-1} \left(x_M^{c_{3M}-1} - x_{IN}^{c_{3M}-1} \right) \\ & + \frac{\kappa_{9M}}{c_{3M}+2} \left(x_M^{c_{3M}+2} - x_{IN}^{c_{3M}+2} \right) + \kappa_{10M} \left(\frac{1}{x_{IN}} - \frac{1}{x_M} \right) \Bigg] d\varphi \, d\nu, \end{aligned} \tag{7.78}$$

where Θ, x_{IN}, x_M and s_{44M}, c_{iM} $(i=1,2)$ are given by Equations (1.15)–(1.17) and (2.13), (2.18), respectively, and ζ, κ_{iM} ($i=2,3$; see Equation (6.13)), κ_{jM} ($j=4,7,9,10$; see Equation (7.14)) have the forms

$$\begin{aligned} \zeta = x_M^{c_{3M}-1} \Bigg\{ & [c_{3M}(c_{1M}+c_{2M}) - 2c_{2M}] \left(\frac{x_{IN}}{x_M} \right)^{c_{3M}-1} \\ & - c_{3M}(c_{1M}+2c_{2M}) \left(\frac{x_M}{x_{IN}} \right)^3 + \frac{c_{2M}(c_{3M}+2)x_M}{x_{IN}} \Bigg\}, \end{aligned}$$

$$\begin{aligned} \kappa_{2M} = & \left[\frac{(c_{1M}+c_{2M})c_{3M}^2}{2} + c_{1M} - 2c_{2M}c_{3M} \right] \left(\frac{p_n}{\zeta x_M^{c_{3M}-1}} \right)^2 \\ & + \frac{1}{s_{44M}} \left\{ \left[\frac{\partial C_2}{\partial \varphi} \left(\frac{p_n}{\zeta x_M^{c_{3M}-1}} \right) \right]^2 + \Theta^2 \left[\frac{\partial C_2}{\partial \nu} \left(\frac{p_n}{\zeta x_M^{c_{3M}-1}} \right) \right]^2 \right\}, \end{aligned}$$

$$\begin{aligned} \kappa_{3M} = & 3(c_{1M}+2c_{2M}) \left(\frac{p_n c_{3M} x_M^3}{2\zeta} \right)^2 \\ & + \frac{1}{2s_{44M}} \left\{ \left[\frac{\partial}{\partial \varphi} \left(\frac{p_n c_{3M} x_M^3}{\zeta} \right) \right]^2 + \Theta^2 \left[\frac{\partial}{\partial \nu} \left(\frac{p_n c_{3M} x_M^3}{\zeta} \right) \right]^2 \right\}, \end{aligned}$$

$$\begin{aligned} \kappa_{4M} = & c_{1M} \left[\frac{p_n x_M (c_{3M}+2)}{2\zeta} \right]^2 \\ & + \frac{c_{3M}+2}{s_{44M}} \left\{ \left[\frac{\partial}{\partial \varphi} \left(\frac{p_n x_M}{2\zeta} \right) \right]^2 + \Theta^2 \left[\frac{\partial}{\partial \nu} \left(\frac{p_n x_M}{2\zeta} \right) \right]^2 \right\}, \end{aligned}$$

$$\kappa_{7M} = [2c_{2M}(1-c_{3M}) - c_{1M}] \left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}} \right) \left(\frac{p_n c_{3M} x_M^3}{2\zeta} \right)$$
$$+ \frac{c_{3M}}{s_{44M}} \frac{\partial}{\partial \varphi} \left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}} \right) \frac{\partial}{\partial \varphi} \left(\frac{p_n x_M^3}{\zeta} \right)$$
$$+ \frac{\Theta^2 c_{3M}}{s_{44M}} \frac{\partial}{\partial \nu} \left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}} \right) \frac{\partial}{\partial \nu} \left(\frac{p_n x_M^3}{\zeta} \right),$$
$$\kappa_{9M} = -\frac{x_M (c_{1M} - c_{2M} c_{3M})(c_{3M}+2)}{2 x_M^{c_{3M}-1}} \left(\frac{p_n}{\zeta} \right)^2$$
$$- \frac{c_{3M}+2}{2 s_{44M}} \frac{\partial}{\partial \varphi} \left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}} \right) \frac{\partial}{\partial \varphi} \left(\frac{p_n x_M}{\zeta} \right)$$
$$- \frac{\Theta^2 (c_{3M}+2)}{2 s_{44M}} \frac{\partial}{\partial \nu} \left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}} \right) \frac{\partial}{\partial \nu} \left(\frac{p_n x_M}{\zeta} \right)$$
$$\kappa_{10M} = -\, c_{3M}(c_{1M} + 2c_{2M})(c_{3M}+2) \left(\frac{p_n x_M^2}{2\zeta} \right)^2$$
$$- \frac{c_{3M}(c_{3M}+2)}{4 s_{44M}} \frac{\partial}{\partial \varphi} \left(\frac{p_n x_M^3}{\zeta} \right) \frac{\partial}{\partial \varphi} \left(\frac{p_n x_M}{\zeta} \right)$$
$$- \frac{\Theta^2 c_{3M}(c_{3M}+2)}{4 s_{44M}} \frac{\partial}{\partial \nu} \left(\frac{p_n x_M^3}{\zeta} \right) \frac{\partial}{\partial \nu} \left(\frac{p_n x_M}{\zeta} \right). \quad (7.79)$$

The coefficients η_2, η_4 and η_4 are given by Equations (6.23) and (7.24), respectively, where ζ in Equations (6.23), (7.24) is given by Equation (7.79). The normal stress p_n is given by Equation (2.33). With regard to Equation (7.71), the coefficient ρ_M in Equation (2.33) is derived as

$$\rho_M = \frac{1}{\zeta} \left[\left(\frac{x_{IN}}{x_M} \right)^{c_{3M}-1} + \frac{c_{3M}}{2} \left(\frac{x_M}{x_{IN}} \right)^3 - \frac{(c_{3M}+2) x_M}{2 x_{IN}} \right]. \quad (7.80)$$

Chapter 8

Strengthening

The analytical model of the micro-strengthening $\sigma_{st} = \sigma_{st}(x_1)$ and the macro-strengthening $\overline{\sigma_{st}}$ results from the following analysis [3, 4, 12, 13, 21]. Figures 8.1 and 8.2 shows the plane $x_2'x_3'$ in the cubic cell (see Figure 1.2) for $x_1 \in \langle 0, a_1 \rangle$ and $x_1 \in \langle a_1, d/2 \rangle$, respectively, where $[x_1, x_2, x_3]$ are coordinates of the point $P \subset x_2'x_3'$. The plane $O'P_1P_2$ with the ellipse E_{23} (see Figure 8.2) represents a cross section of the ellipsoid inclusion in the plane $x_2'x_3'$. With regard to Figures (8.1), (8.2), the goniometric functions in Equations (1.8)–(1.17) have the forms

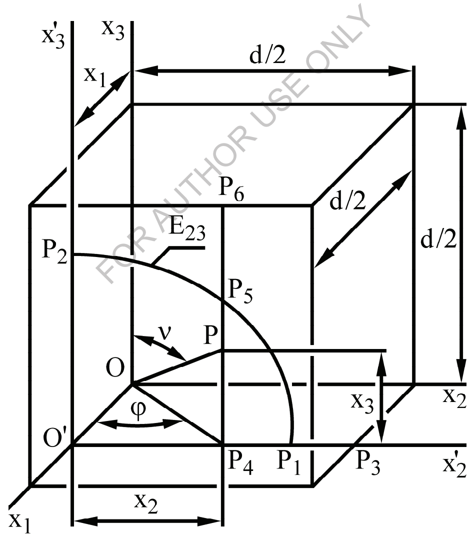


Figure 8.1: The plane $x_2'x_3'$ in the cubic cell (see Figure 1.2) for $x_1 \in \langle 0, a_1 \rangle$, where $[x_1, x_2, x_3]$ are coordinates of the point $P \subset x_2'x_3'$. The plane $O'P_1P_2$ with the ellipse E_{23} represents a cross section of the ellipsoid inclusion in the plane $x_2'x_3'$ (see Figure 1.2).

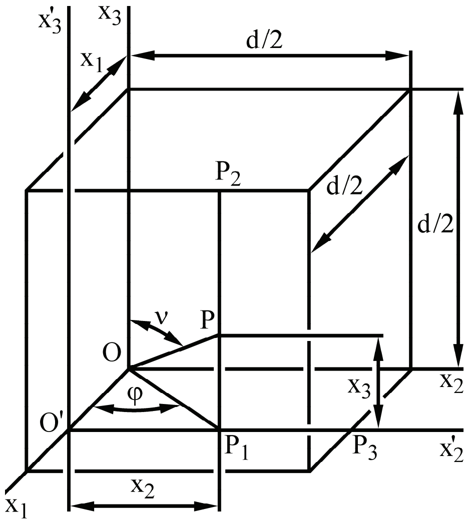


Figure 8.2: The plane $x_2'x_3'$ in the cubic cell (see Figure 1.2) for $x_1 \in \langle a_1, d/2 \rangle$, where $[x_1, x_2, x_3]$ are coordinates of the point $P \subset x_2'x_3'$.

$$\sin\varphi = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}, \quad \cos\varphi = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \quad \tan\varphi = \frac{1}{\cot} = \frac{x_2}{x_1},$$

$$\sin\nu = \sqrt{\frac{x_1^2 + x_2^2}{x_1^2 + x_2^2 + x_3^2}}, \quad \cos\nu = \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}, \quad x_n = \frac{x_3}{\cos\theta}, \tag{8.1}$$

where $\cos\theta$ is given by Equation (1.13). With regard to Equation (1.2), the parameters b_2, b_3 of the ellipse E_{23} along the axes x_2', x_3', respectively, are derived as (see Figure 8.1)

$$b_2 = O'P_1 = \frac{a_2\sqrt{a_1^2 - x_1^2}}{a_1}, \quad b_3 = O'P_2 = \frac{a_3\sqrt{a_1^2 - x_1^2}}{a_1}, \tag{8.2}$$

and then we get

$$b_4 = P_4P_5 = \frac{a_3\sqrt{b_2^2 - x_2^2}}{a_2}. \tag{8.3}$$

The micro-strengthening $\sigma_{st} = \sigma_{st}(x_1)$ represents a stress along the axis x_1, which is homogeneous at each point of the plane $x_2'x_3'$ with the area $S = d^2/4$, i.e., $\sigma_{st} \neq f(x_2, x_3)$.

If $x_1 \in \langle 0, a_1 \rangle$, then the elastic energy surface density W_{st}, which is induced by σ_{st} and accumulated within the area $S_{IN} = \pi b_2 b_3/4$ of the plane $O'P_1P_2$ and within the area $S_M = (d/2)^2 - S_{IN}$ of the plane $x_2'x_3'$ (see Figure 8.1), has the form

$$W_{st} = \omega \sigma_{1st}^2, \tag{8.4}$$

where σ_{1st} is related to $x_1 \in \langle 0, a_1 \rangle$. The coefficient ω is derived as

$$\omega = \frac{1}{8}\left[\pi b_2 b_3 \left(\frac{1}{E_{IN}} - \frac{1}{E_M}\right) + \frac{d^2}{E_M}\right], \tag{8.5}$$

where E_{IN} and E_M is Young's modulus for the ellipsoidal inclusion and the matrix, respectively. The elastic energy surface density W_{1S}, which is induced by the stress $\sigma_1 = \sigma_1(x_1)$ (see Equations (3.22), (4.22), (4.33), (4.44), (4.55), (5.20), (5.31), (5.43), (5.54), (6.20), (6.31), (6.42), (6.53), (7.21), (7.32), (7.43), (7.54), (7.65), (7.76)), has the form

$$\begin{aligned}
W_{1S} &= \frac{1}{2}\left(\frac{W_{INS}}{E_{IN}} + \frac{W_{1MS}}{E_M}\right), \\
W_{INS} &= \int_0^{b_2}\left(\int_0^{b_4} \sigma_1^2 \, dx_3\right) dx_2, \\
W_{1MS} &= \int_0^{b_2}\left(\int_{b_4}^{d/2} \sigma_1^2 \, dx_3\right) dx_2 + \int_{b_2}^{d/2}\left(\int_0^{d/2} \sigma_1^2 \, dx_3\right) dx_2, \quad x_1 \in \langle 0, a_1 \rangle.
\end{aligned} \tag{8.6}$$

The micro-strengthening $\sigma_{1st} = \sigma_{1st}(x_1)$ for $x_1 \in \langle 0, a_1 \rangle$, which results from the condition $W_{st} = W_{1S}$ [3, 4, 12, 13, 21], is derived as

$$\sigma_{1st} = \sqrt{\frac{1}{2\omega}\left(\frac{W_{INS}}{E_{IN}} + \frac{W_{1MS}}{E_M}\right)}, \quad x_1 \in \langle 0, a_1 \rangle. \tag{8.7}$$

If $x_1 \in \langle a_1, d/2 \rangle$, then the elastic energy surface density W_{st}, which is induced by σ_{st} and accumulated within the area $S_M = d^2/4$ of the plane $x_2'x_3'$ (see Figure 8.2), has the form

$$W_{st} = \frac{\sigma_{2st}^2 d^2}{8 E_M}, \tag{8.8}$$

where σ_{2st} is related to $x_1 \in \langle a_1, d/2 \rangle$. Similarly, we get

$$W_{2S} = \frac{W_{2MS}}{2E_M}, \quad W_{2MS} = \int_0^{d/2} \int_0^{d/2} \sigma_1^2 \, dx_2 \, dx_3, \quad x_1 \in \left\langle a_1, \frac{d}{2} \right\rangle. \tag{8.9}$$

With regard to the condition $W_{st} = W_{2S}$ [3, 4, 12, 13, 21], we get

$$\sigma_{2st} = \frac{2\sqrt{W_{2S}}}{d}. \tag{8.10}$$

Finally, the macro-strengthening $\overline{\sigma_{st}}$ is derived as [3, 4, 12, 13, 21]

$$\overline{\sigma_{st}} = \frac{2}{d} \left(\int_0^{a_1} \sigma_{1st} \, dx_1 + \int_{a_1}^{d/2} \sigma_{2st} \, dx_1 \right). \tag{8.11}$$

If $\alpha_{IN} < \alpha_M$ or $\alpha_{IN} > \alpha_M$, the strengthening exhibits a resistive effect against compressive or tensile mechanical loading, respectively.

The macro-strengthening $\overline{\sigma_{st}} = \overline{\sigma_{st}}\,(v, a_1, a_2, a_3)$ is a function of the inclusion volume fraction v_{IN} and the dimensions a_1, a_2, a_3 of the ellipsoidal inclusion. In case of a real inclusion-matrix composite, such values of the microstructural parameters v_{IN}, a_1, a_2, a_3 can be numerically determined to result in a maximum value of $|\overline{\sigma_{st}}|$.

Chapter 9

Crack Formation

The analytical model of the crack formation in the matrix results from the following analysis [3, 4, 5, 19]–[22]. Figures 9.1, 9.3 show the ellipse E_{123} in the plane $x_{12}x_3$ of the cubic cell (see Figures (1.4), (1.5)), where $a_{12} = O4$, $x_{122} = O5$ are given by Equations (1.7), (1.11), and $a_3 = O3$.

With regard to the plane $x_{12}x_3$ for $\varphi \in \langle 0, \pi/2 \rangle$ (see Figures 1.4, 1.5), the elastic energy density $w = w(x_n, \varphi, \nu)$ (see Equations (3.23), (3.34), (4.23), (4.34), (4.45), (4.56), (5.22), (5.33), (5.44), (5.55), (6.21), (6.32), (6.43), (6.54), (6.63), (7.22), (7.33), (7.44), (7.55), (7.66), (7.77)) is determined as a function of the coordinates x_n, $\nu \in \langle 0, \pi/2 \rangle$ (see Equations (1.6)–(1.17)). The elastic energy density $w = w(x_{12}, \varphi, x_3, a_1, a_2, a_3, v_{IN})$ as a function of the coordinates x_{12}, x_3 is determined by the following transformations

$$x_n = \frac{x_3}{\cos\theta}, \quad \sin\nu = \frac{x_{12}}{\sqrt{x_{12}^2 + x_3^2}}, \quad \cos\nu = \frac{x_3}{\sqrt{x_{12}^2 + x_3^2}}, \quad \tan\nu = \frac{1}{\cot\nu} = \frac{x_{12}}{x_3}, \quad (9.1)$$

where $\cos\theta$ is given by Equation (1.13).

Matrix. The curve integral W_{cM} of $w_M = w_M(x_{12}, \varphi, x_3, a_1, a_2, a_3, v_{IN})$ along the abscissa P_1P_2 (see Figure 9.1) in the plane $x_{12}x_3$ of the matrix (see Figures 1.4, 1.5) has the form

$$W_{cM} = \int_{P_1P_2} w_M \, dx_3 = \int_0^{d/2} w_M \, dx_3. \quad (9.2)$$

Let $f_{12M} = f_{12M}(x_{12}, \varphi, a_1, a_2, a_3, v_{IN})$ represent a decreasing function of the variable $x_{12} \in \langle a_{12}, x_{0M} \rangle$, which describe a shape of the matrix crack in the plane $x_{12}x_3$ (see Figure 1.4), where $\varphi \in \langle 0, \pi/2 \rangle$, a_1, a_2, a_3, v_{IN} are parameters of this decreasing function. As presented in [3, 4, 5, 19]–[22], we get

$$\frac{\partial f_{12M}}{\partial x_{12}} = -\frac{\sqrt{W_{cM}^2 - \vartheta_M^2}}{\vartheta_M}, \quad (9.3)$$

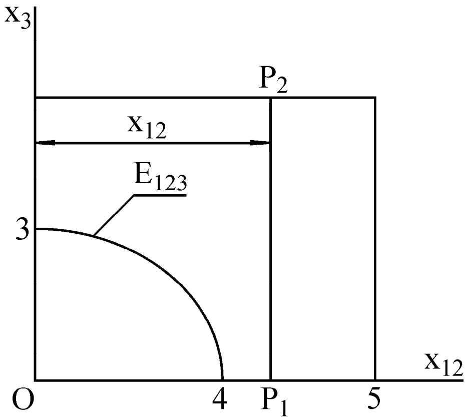


Figure 9.1: The ellipse E_{123} and the abscissa P_1P_2 in the plane $x_{12}x_3$ of the cubic cell (see Figures (1.4), (1.5)), where $a_{12} = O4$, $x_{122} = O5$ are given by Equations (1.7), (1.11), and $a_3 = O3$.

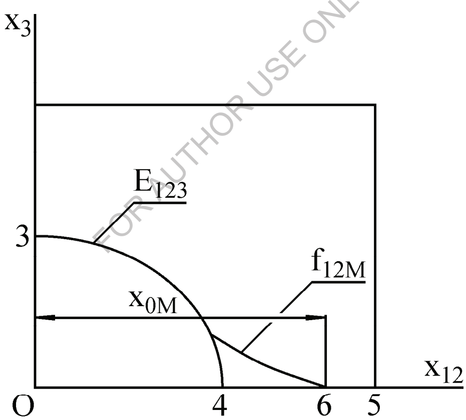


Figure 9.2: The decreasing function $f_{12M} = f_{12M}(x_{12}, \varphi, a_1, a_2, a_3, v_{IN})$ of the variable $x_{12} \in \langle a_{12}, x_{0M} \rangle$, which describes a shape of the matrix crack in the plane $x_{12}x_3$ (see Figure 1.4) for $a_{12} > a_{12M}^{(IC)}$ or $a_{12} > a_{12M}^{(TC)}$ (see Equations (9.8), (9.9)), where $x_{0M} = x_{0M}(\varphi)$ defnes a position of the crack tip in the matrix, and $\varphi \in \langle 0, \pi/2 \rangle$, a_1, a_2, a_3, v_{IN} are parameters of this decreasing function.

where ϑ_M is energy per unit length in the matrix. In case of intercrystalline crack formation, we get

$$\vartheta_M = \frac{K_{ICM}^2}{E_M}, \tag{9.4}$$

where K_{ICM} is fracture toughness of the matrix. In case of transcrystalline crack formation, we get

$$\vartheta_M = \vartheta_{gbM}, \tag{9.5}$$

where the energy ϑ_{gbM} per unit length is related to the inter-atomic bonding of boundaries of crystalline grain in the matrix.

As presented in [3, 4, 5, 19]–[22], the condition

$$(W_{cM})_{x_{12}=a_{12}} - \vartheta_M = 0, \tag{9.6}$$

is a transcendental equation with the variable a_{12} and the parameters $\varphi \in \langle 0, \pi/2 \rangle$, a_1, a_2, a_3, v_{IN} (see Figure 1.4.

The roots $a_{12M}^{(IC)} = a_{12M}^{(IC)}(\varphi, a_1, a_2, a_3, v_{IN})$ and $a_{12M}^{(TC)} = a_{12M}^{(TC)}(\varphi, a_1, a_2, a_3, v_{IN})$ (see Equation (1.7)) of Equation (9.3) for ϑ_M, which is given by Equations (9.4) and (9.5), represents such a dimension of the ellipsoidal inclusion along the axis $x_{12} \subset x_1x_2$ (see Figures 1.4, 1.5), which is critical with respect to the intercrystalline and transcrystalline crack formation in the plane x_1x_2, respectively. Accordingly, if $a_{12M}^{(IC)} > a_{12M}^{(TC)}$ or $a_{12M}^{(IC)} < a_{12M}^{(TC)}$, then the intercrystalline or transcrystalline matrix crack is formed in the plane x_1x_2, respectively.

Let the function $a_{12M}^{(X)} = a_{12M}^{(X)}(\varphi, a_1, a_2, a_3, v_{IN})$ (X=IC,TC) of the variable $\varphi \in \langle 0, \pi/2 \rangle$ exhibit the minimum $a_{minM}^{(X)}$ for $\varphi = \varphi_{minM}^{(X)}$. The critical dimension $a_{minM}^{(X)} = a_{minM}^{(X)}(a_1, a_2, a_3, v_{IN})$ (X=IC,TC) along the axis $x_{12} \subset x_1x_2$ (see Figures 1.4, 1.5) defnes a limit state with respect to the formation of the intercrystalline matrix crack (X=IC) and the transcrystalline matrix crack (X=TC) in the plane x_1x_2 at the microstructural parameters a_1, a_2, a_3, v_{IN} (see Equation (1.1)). Accordingly, if $a_{12} > a_{12M}^{(X)}$ (X=IC,TC), the condition [3, 4, 5, 19]–[22]

$$W_{cM} - \vartheta_M = 0, \quad a_{12} > a_{12M}^{(X)}, \quad X = IC, TC \tag{9.7}$$

represents a transcendental equation with the variable x_{12} and with the root $x_{0M} = x_{0M}(\varphi, a_2, a_3, v_{IN})$, which defnes a position of the crack tip in the matrix (see Figure 9.2). Consequently, the decreasing function $f_{12M} = f_{12M}(x_{12}, \varphi, a_1, a_2, a_3, v_{IN})$ with the variable $x_{12} \in \langle a_{12}, x_{0M} \rangle$ and with the parameters $\varphi \in \langle 0, \pi/2 \rangle$, a_1, a_2, a_3, v_{IN} (see Figures 1.4, 1.5), which describes a shape of the matrix crack in the plane $x_{12}x_3$ for $a_{12} > a_{12M}^{(X)}$ ((X=IC,TC)), has the form [3, 4, 5, 19]–[22]

$$f_{12M} = \frac{1}{\vartheta_M}\left[C_M - \int\left(\sqrt{W_{cM}^2 - \vartheta_M^2}\right)dx_{12}\right], \quad x_{12} \in \langle a_{12}, x_{0M}\rangle, \tag{9.8}$$

where $C_M = C_M(\varphi, a_1, a_2, a_3, v_{IN})$ is derived as [3, 4, 5, 19]–[22]

$$C_M = \left[\int\left(\sqrt{W_{cM}^2 - \vartheta_M^2}\right)dx_{12}\right]_{x_{12}=x_{0M}}. \tag{9.9}$$

Inclusion. The curve integral W_{cIN} of $w_{IN} = w_{IN}(x_{12}, \varphi, x_3, a_1, a_2, a_3, v_{IN})$ along the abscissa P_1P_2 (see Figure 9.3) in the plane $x_{12}x_3$ of the ellipsoidal inclusion (see Figures 1.4, 1.5) has the form

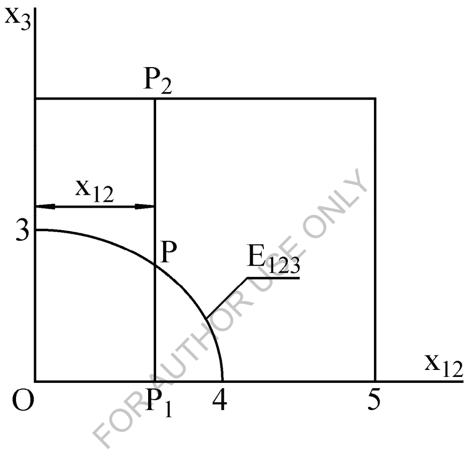


Figure 9.3: The ellipse E_{123} and the abscissa P_1PP_2 in the plane $x_{12}x_3$ of the cubic cell (see Figures (1.4), (1.5)), where $a_{12} = O4$, $x_{122} = O5$ are given by Equations (1.7), (1.11), and $a_3 = O3$.

$$W_{cIN} = \int_{P_1P} w_{IN}\,dx_3 + \int_{PP_2} w_M\,dx_3 = \int_0^{b_1} w_{IN}\,dx_3 + \int_{b_1}^{d/2} w_M\,dx_3, \tag{9.10}$$

where $a_{12} = O4$ (see Equation (1.7)), $a_3 = O3$, and b_1 is derived as (see Equation (1.2))

$$b_1 = P_1P = \frac{a_3\sqrt{a_{12}^2 - x_{12}^2}}{a_{12}}, \quad x_{12} \in \langle 0, a_{12}\rangle. \tag{9.11}$$

With regard to the intercrystalline and transcrystalline inclusion cracks (see Figure 9.4), the sign '-' and the subscript M in Equations (9.3) and (9.3)–(9.7) are replaced by the sign '+' and the subscript IN, respectively.

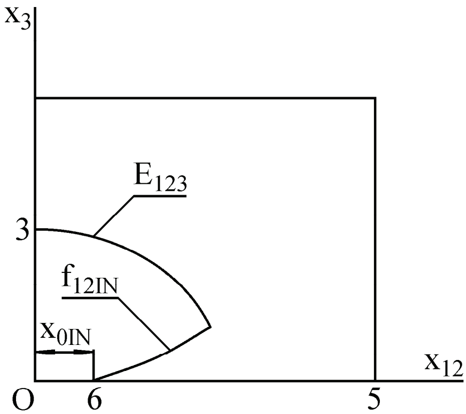


Figure 9.4: The increasing function $f_{12IN} = f_{12IN}(x_{12}, \varphi, a_1, a_2, a_3, v_{IN})$ of the variable $x_{12} \in \langle a_{12}, x_{0IN} \rangle$, which describes a shape of the inclusion crack in the plane $x_{12}x_3$ (see Figure 1.4) for $a_{12} > a_{12IN}^{(IC)}$ or $a_{12} > a_{12IN}^{(TC)}$ (see Equations (9.8), (9.9)), where $x_{0IN} = x_{0IN}(\varphi)$ defnes a position of the crack tip in the inclusion, and $\varphi \in \langle 0, \pi/2 \rangle$ is a parameter of this increasing function.

Consequently, the increasing function $f_{12IN} = f_{12IN}(x_{12}, \varphi, a_1, a_2, a_3, v_{IN})$ with the variable $x_{12} \in \langle a_{12}, x_{0IN} \rangle$ and with the parameters $\varphi \in \langle 0, \pi/2 \rangle$, a_1, a_2, a_3, v_{IN} (see Figures 1.4, 1.5), which describes a shape of the inclusion crack in the plane $x_{12}x_3$ for $a_{12} > a_{12IN}^{(IC)}$ or $a_{12} > a_{12IN}^{(TC)}$, has the form [3, 4, 5, 19]–[22]

$$f_{12IN} = \frac{1}{\vartheta_{IN}} \left[\int \left(\sqrt{W_{cIN}^2 - \vartheta_{IN}^2} \right) dx_{12} - C_{IN} \right], \quad x_{12} \in \langle a_{12}, x_{0IN} \rangle, \tag{9.12}$$

where $C_{IN} = C_{IN}(\varphi, a_1, a_2, a_3, v_{IN})$ is derived as [3, 4, 5, 19]–[22]

$$C_{IN} = \left[\int \left(\sqrt{W_{cIN}^2 - \vartheta_{IN}^2} \right) dx_{12} \right]_{x_{12} = x_{0IN}}. \tag{9.13}$$

Chapter 10

Appendix

Cramer's Rule. The system of n linear algebraic equations is derived as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1, \\ a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_2, \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n. \end{aligned} \tag{10.1}$$

The root x_i ($i = 1, \ldots,$ n) is determined by Cramer's rule [23]

$$x_i = \frac{D_i^{(n)}}{D^{(n)}}, \quad i = 1, \ldots, n, \tag{10.2}$$

where the determinant $D^{(n)}$ with n rows and n columns has the form

$$\begin{aligned} D^{(n)} &= \begin{vmatrix} a_{11}, & a_{12}, & \ldots & a_{1n} \\ a_{21}, & a_{22}, & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}, & a_{n2}, & \ldots & a_{nn} \end{vmatrix} \\ &= \sum_{i=1}^{n} (-1)^{1+i} a_{1i} D_{1i}^{(n-1)} = \sum_{i=1}^{n} (-1)^{1+i} a_{i1} D_{i1}^{(n-1)}. \end{aligned} \tag{10.3}$$

The subdeterminant $D_i^{(n)}$ is created from $D^{(n)}$, i.e., the i-th column of $D^{(n)}$ is replaced by

$$\left.\begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{matrix}\right\} n \text{ rows.} \tag{10.4}$$

Similarly, the subdeterminant $D_{ij}^{(n-1)}$ ($i, j = 1, \ldots,$ n) with $(n-1)$ rows and $(n-1)$ columns is created from $D^{(n)}$, i.e., the i-th row and the j-th column of $D^{(n)}$ are omitted. If $n = 2$, then we get

$$D^{(2)} = \begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \tag{10.5}$$

Consequently, if $n = 3$, then we get

$$\begin{aligned} D^{(4)} &= \begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix} \\ &= a_{11} \begin{vmatrix} a_{22}, & a_{23} \\ a_{32}, & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21}, & a_{23} \\ a_{31}, & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21}, & a_{22} \\ a_{31}, & a_{32} \end{vmatrix}. \end{aligned} \tag{10.6}$$

Integrals. The derivatives of the functions $f = x^{\lambda}$, $f = \ln x$ and the constant C are derived as [23]

$$\left(x^{\lambda}\right)' = \lambda x^{\lambda-1}, \quad (\ln x)' = \frac{1}{x}, \ C' = 0, \tag{10.7}$$

The indefinite integrals of $f = x^{\lambda}$, $f = \ln x$ and the constant C have the forms [23]

$$\int x^{\lambda}\,dx = \frac{x^{\lambda}+1}{\lambda+1}, \quad \lambda \neq -1; \int \frac{dx}{x} = \ln x, \quad \int C\,dx = Cr. \tag{10.8}$$

In case of the product fg of the functions $f = f(x)$, $g = g(x)$, we get [23]

$$(fg)' = f'g + fg'. \tag{10.9}$$

and then the integral of fg has the form [23]

$$\int f'g\,dx = fg - \int fg'\,dx. \tag{10.10}$$

With regard to Equation (10.17), the following integrals are derived as [23]

$$\begin{aligned} &\int x^{\lambda} \ln x\,dx = \frac{x^{\lambda+1}}{\lambda+1} \ln x - \int \frac{x^{\lambda+1}}{\lambda+1} \times \frac{1}{x}\,dx = \frac{x^{\lambda+1}}{\lambda+1} \ln x - \frac{1}{\lambda+1} \int x^{\lambda}\,dx \\ &\quad = \frac{x^{\lambda+1}}{\lambda+1}\left(\ln x - \frac{1}{\lambda+1}\right), \quad \lambda \neq -1, \\ &\int \ln x\,dx = \int 1 \times \ln x\,dx = x \ln x - \int x \times \frac{1}{x}\,dx = x \ln x - \int 1 \times dx = x\left(\ln x - 1\right), \\ &\int x^{\lambda} \ln^2 x\,dx = \frac{1}{\lambda+1}\left[x^{\lambda+1} \ln^2 x - 2\int x^{\lambda} \ln x\,dx\right] \end{aligned}$$

$$= \frac{x^{\lambda+1}}{\lambda+1}\left[\left(\ln x - \frac{1}{\lambda+1}\right)^2 + \frac{1}{(\lambda+1)^2}\right], \quad \lambda \neq -1. \qquad (10.11)$$

Let $F = F(x)$ be a primitive function of $f = f(x)$ in the interval $x \in \langle a,b \rangle$, i.e., $f = dF/dx$. The definite integral $\int_a^b f\,dx$ is defined by Newton-Leibniz's formula [23], which has the form

$$\int_a^b f\,dx = F(b) - F(a). \qquad (10.12)$$

Wronskian's Method. The differential equation (4.3) with a non-zero right-hand side [23] is derived as

$$\frac{\partial^2 u_n}{\partial x^2} + \frac{2}{x_n}\frac{\partial u_n}{\partial x_n} - \frac{2u_n}{x_n^2} = g, \quad g = \sum_{i=1}^{3} C_i x^{\kappa_i - 2}, \qquad (10.13)$$

where the integration constants C_1, C_2, C_3 are determined by the boundary conditions in Section 2.3. If $g = 0$, we get

$$\frac{\partial^2 u_n}{\partial x^2} + \frac{2}{x_n}\frac{\partial u_n}{\partial x_n} - \frac{2u_n}{x_n^2} = 0. \qquad (10.14)$$

If $u_n = x^{\lambda}$, then the solutions u_{1n}, u_{2n} of Equation (10.24) have the forms

$$u_{1n} = x_n, \quad u_{2n} = \frac{1}{x_n^2}. \qquad (10.15)$$

The solution u_n of Equation (10.22) is derived as [23]

$$u_n = \sum_{i=1}^{2} a_i u_{in}, \quad a_i = \int \frac{W_i^{(2)}}{W^{(2)}}\,dx_n, \quad i = 1,2. \qquad (10.16)$$

Wronskian's determinants $W^{(2)}$, $W_i^{(2)}$ $(i = 1,2)$ with 2 rows and 2 columns are have the forms [23]

$$W^{(2)} = \begin{vmatrix} u_{1n}, & u_{2n} \\ \dfrac{\partial u_{1n}}{\partial x_n}, & \dfrac{\partial u_{2n}}{\partial x_n} \end{vmatrix}, \quad W_1^{(2)} = \begin{vmatrix} 0, & u_{2n} \\ \mathrm{g}, & \dfrac{\partial u_{2n}}{\partial x_n} \end{vmatrix}, \quad W_2^{(2)} = \begin{vmatrix} u_{1n}, & 0 \\ \dfrac{\partial u_{1n}}{\partial x_n}, & \mathrm{g} \end{vmatrix}. \qquad (10.17)$$

The determinant $W_i^{(2)}$ $(i = 1,2)$ is created from $W^{(2)}$, i.e., the i-th column of $W^{(2)}$ is replaced by the following one [23]

$$\left.\begin{matrix} 0 \\ g \end{matrix}\right\} \text{2 rows.} \tag{10.18}$$

Let $f_1, \ldots, f_n$ represent n solutions of a differential equation of the n-th rank with zero right-hand side. Let the functions $f_1, \ldots, f_n$ of the variable x exhibit continuous derivatives to the $(n-1)$-th degree. The solution of this differential equation with a non-zero right-hand side (i.e., $g \neq 0$) is derived as [23]

$$f = \sum_{i=1}^{n} a_i f_i, \quad a_i = \int \frac{W_i^{(n)}}{W^{(n)}}\, dx. \tag{10.19}$$

With respect to $f_1, \ldots, f_n$, Wronskian's determinant $W^{(n)}$ $(i = 1, \ldots, \mathrm{n})$ with n rows and n columns have the form [23]

$$W^{(n)} = \begin{vmatrix} f_1, & f_2, & \cdots & f_n \\ \frac{\partial f_1}{\partial x}, & \frac{\partial f_2}{\partial x}, & \cdots & \frac{\partial f_n}{\partial x} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^{n-1} f_1}{\partial x^{n-1}}, & \frac{\partial^{n-1} f_2}{\partial x^{n-1}}, & \cdots & \frac{\partial^{n-1} f_n}{\partial x^{n-1}} \end{vmatrix}, \tag{10.20}$$

where $W_i^{(n)}$ $(i = 1, \ldots, \mathrm{n})$ with n rows and n columns is created from $W^{(n)}$, i.e., the i-th column of $W^{(n)}$ is replaced by the following one [23]

$$\left.\begin{matrix} 0 \\ 0 \\ \vdots \\ g \end{matrix}\right\} n \text{ rows.} \tag{10.21}$$

Numerical Determination. Numerical values of the thermal stresses in a real matrix-inclusion composite include integrals and derivatives, which are determined by a programming language. If $f = f(x)$, then a numerical value of the derivative $\partial f / \partial x$ is determined by [23]

$$\frac{\partial f}{\partial x} \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{10.22}$$

In case of the angles φ, ν (see Figure 1.4), the step $\Delta x = \Delta\varphi = \Delta\nu = 10^{-6}$ [deg] is sufficient [3, 4, 5, 19]–[22].

Let F represent a definite integral of the function $f = f(\varphi, \nu)$ with the variables $\varphi, \nu \in \langle 0, \pi/2 \rangle$. Let n, m be integral parts of the real numbers $\pi/(2\Delta\varphi)$, $\pi/(2\Delta\nu)$

[3, 4, 5, 19]–[22], respectively. Numerical values of the definite integral F are determined by the following formula [23], [3, 4, 5, 19]–[22]

$$F = \int_0^{\pi/2} \int_0^{\pi/2} f(\varphi, \nu) \, d\varphi \, d\nu \approx \sum_{j=0}^{m} \left(\sum_{i=0}^{n} f(i \times \Delta\varphi; j \times \Delta\nu) \Delta\varphi \right) \Delta\nu, \tag{10.23}$$

where the steps $\Delta\varphi = \Delta\nu = 0.1$ [deg] are sufficient. Finally, the average numerical value $\overline{f}$ of the function $f = f(\varphi, \nu)$ with the variables $\varphi, \nu \in \langle 0, \pi/2 \rangle$ is determined by the following formula [23]

$$\overline{f} = \left(\frac{2}{\pi}\right)^2 \int_0^{\pi/2} \int_0^{\pi/2} f(\varphi, \nu) \, d\varphi \, d\nu \approx \left(\frac{2}{\pi}\right)^2 \sum_{j=0}^{m} \left(\sum_{i=0}^{n} f(i \times \Delta\varphi; j \times \Delta\nu) \Delta\varphi \right) \Delta\nu. \tag{10.24}$$

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Index