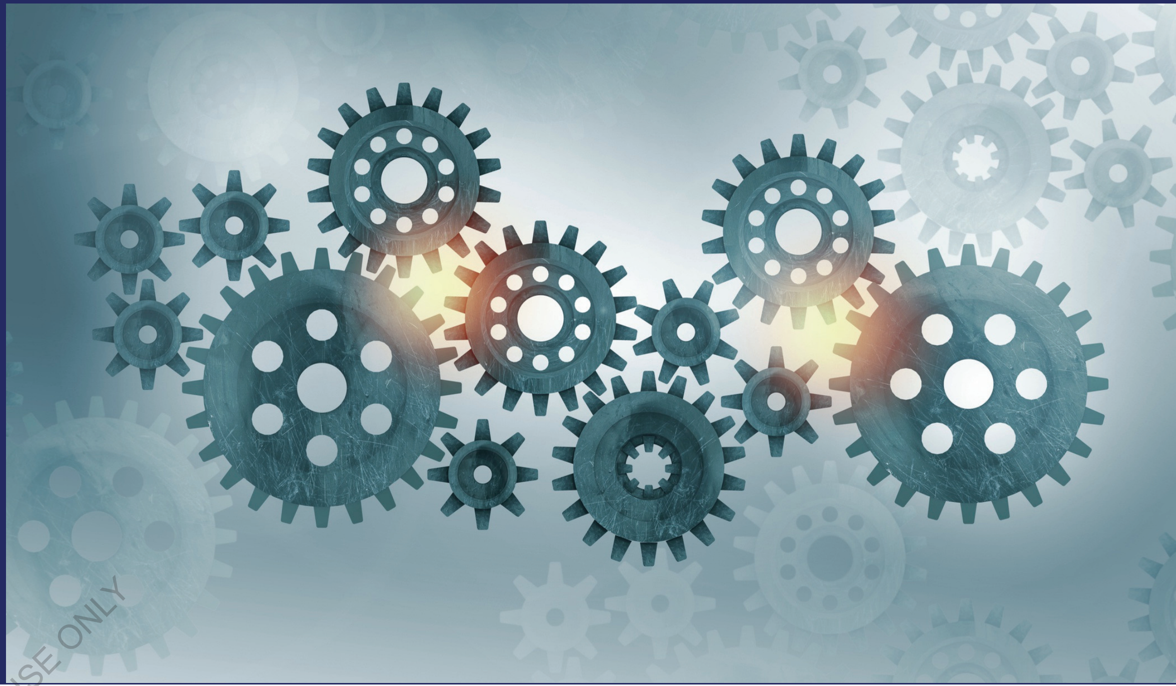
This book presents original mathematical models of thermal-stress-field interactions in composite materials, along with mathematical models of thermal-stress induced micro/macro-strengthening and intercrystalline or transcrystalline crack formation. The mathematical determination results from mechanics of an isotropic elastic continuum. The materials consist of an isotropic matrix with isotropic ellipsoidal inclusions. The thermal stresses are a consequence of different thermal expansion coefficients of the material components. The interactions are determined by suitable mathematical boundary conditions, as well as by a suitable iteration method. The mathematical models include microstructural parameters of a real matrix-inclusion composite, and are applicable to composites with ellipsoidal inclusions of different morphology (e.g., dual-phase steel, martensitic steel). In case of a real matrix-inclusion composite, such numerical values of the microstructural parameters can be determined, which result in maximum values of the strengthening, and which define limit states with respect to the crack formation.



Dr. Ladislav Ceniga, DSc. (Institute of Materials Research, Slovak Academy of Sciences, Kosice, Slovak Republic) works on mathematical models of stresses in composites.

Ladislav Ceniga

Thermal-Stress-Field Interactions in Composite Materials III

Iteration Method

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Publisher:
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120 High Road, East Finchley, London, N2 9ED, United Kingdom
Str. Armeneasca 28/1, office 1, Chisinau MD-2012, Republic of Moldova, Europe
Printed at: see last page
ISBN: 978-620-7-80520-4

This book was reviewed by the following reviewers:

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Táto kniha je venovaná s láskou mojim najdrahším rodičom a starým rodičom.

This book is dedicated with love to my dearest parents and grandparents.

Contents

Introduction

This book[1] presents original mathematical models of thermal-stress-field interactions in composite materials (see Chapters 3–7), along with mathematical models of thermal-stress induced micro-/macro-strengthening (see Chapter 8) and thermal-stress induced intercrystalline or transcrystalline crack formation (see Chapter 9). The materials consist of an isotropic matrix with isotropic ellipsoidal inclusions. These stresses originate during a cooling process, and are a consequence of different thermal expansion coefficients of the matrix and ellipsoidal inclusions.

The mathematical models are determined for a suitable model system. The model system is required to correspond to real isotropic matrix-inclusion composites. The thermal stresses are derived within a suitable coordinate system. The coordinate system is required to correspond to a shape of the ellipsoidal inclusions (see Chapter 1).

The mathematical determination results from mechanics of an isotropic elastic continuum (see Section 2.1), and result in different mathematical solutions for the thermal stresses (see Sections 3.1, 4.1, 5.1, 6.1, 7.1). Due to these different mathematical solutions, the principle of minimum elastic energy is considered (see Section 2.4).

The mathematical models of the thermal-stress-field interactions, which are determined by an iteration method (see Section 2.3.1), along with the mathematical models of the thermal-stress induced micro-/macro-strengthening and crack formation (see Chapters 8, 9), include microstructural parameters of a real matrix-inclusion composite, i.e., the inclusion dimensions a_1, a_2, a_3, the inclusion volume fraction v_{IN}, as well as the inter-inclusion distance $d = d(a_1, a_2, a_3, v_{IN})$ (see Chapter 1).

The iteration method results from mathematical boundary conditions for the first iteration and for the $(N+1)$-th iteration ($N = 1,2,3,\dots$), as well as from such a mathematical procedure, when the $(N+1)$-th iteration considers mathematical results of the N-th iteration, as well as mathematical results of the 1-st iteration (see Section 2.3.1).

[1]This book was reviewed by the following reviewers:

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Consequently, the mathematical models are applicable to composites with ellipsoidal inclusions of different morphology (see Chapter 1), i.e., $a_1 \approx a_2 \approx a_3$ (dual-phase steel), $a_1 \gg a_2 \approx a_3$ (martensitic steel).

In case of a real matrix-inclusion composite, such numerical values of the microstructural parameters can be determined, which result in maximum values of the micro- and macro-strengthening (see Chapter 8), and which define limit states with respect to the intercrystalline or transcrystalline crack formation in the matrix and the ellipsoidal inclusion (see Chapter 9). This numerical determination is performed by a programming language. The mathematical procedures in this book are analysed in Appendix.

Chapter 1

Matrix-Inclusion Composite

Figure 1.1 shows a model system, corresponding to real matrix-inclusion composites, which is considered within the mathematical models of the thermal stresses. This model system consists of an infinite isotropic matrix and isotropic ellipsoidal inclusions with the dimensions a_1, a_2, a_3 and the inter-inclusion distance d along the axes x_1, x_2, x_3 of the Cartesian system $(Ox_1x_2x_3)$, respectively, where O represents a centre of the ellipsoidal inclusion.

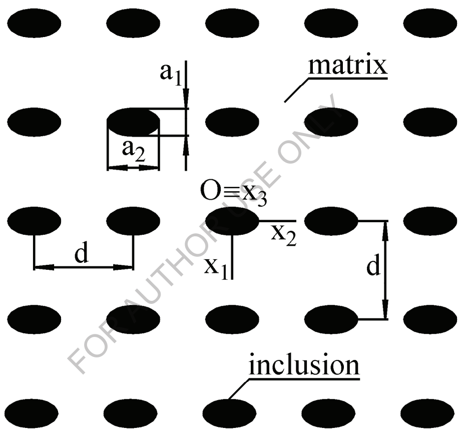


Figure 1.1: The matrix-inclusion system with an infinite isotropic matrix and isotropic ellipsoidal inclusions with the dimensions a_1, a_2, a_3 and the inter-inclusion distance d along the axes x_1, x_2, x_3 of the Cartesian system $(Ox_1x_2x_3)$, respectively, where O represents a centre of the ellipsoidal inclusion.

As presented in [1]–[22], the thermal stresses are determined in the cubic cells with the dimension d along the axes x_1, x_2, x_3 and with central ellipsoidal inclusions (see Figure 1.2). Due to the infinite matrix, the thermal stresses, which are determined for one of the cubic cells, are identical with those, which are determined for any of the cubic cells [1]–[22]. With regard to the volume $V_{IN} = 4\pi a_1, a_2 a_3$ [23] and $V_C = d^3$ of the ellipsoidal inclusion and the cubic cell, the inter-inclusion distance d as a function of the inclusion volume fraction v_{IN} is derived as

$$v_{IN} = \frac{V_{IN}}{V_C} = \frac{4\pi a_1 a_2 a_3}{3d^3} \in \left(0, \frac{\pi}{6}\right), \quad d = \left(\frac{4\pi a_1 a_2 a_3}{3v_{IN}}\right)^{1/3}, \tag{1.1}$$

where the value $v_{INmax} = \pi/6$ results from the condition $a_i \to d/2$ ($i = 1,2,3$). Accordingly, the thermal stresses are functions of the material parameters a_1, a_2, a_3, v_{IN}, d.

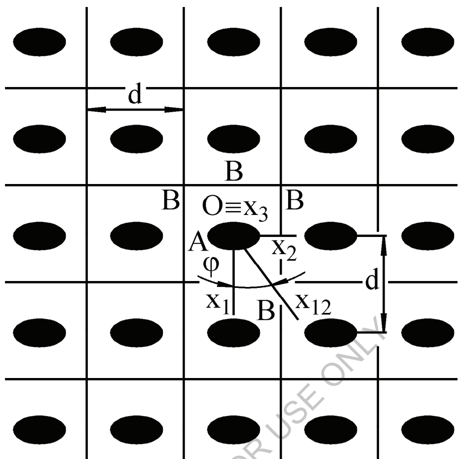


Figure 1.2: The cubic cells with the dimension d along the axes x_1, x_2, x_3 of the Cartesian system ($Ox_1x_2x_3$) and with the plane $x_{12}x_3$, where O represents a centre of the ellipsoidal inclusion, and ($x_{12} \subset x_1x_2$, $x_{12}x_3 \perp x_1x_2$. The thermal stresses in the cell A and the neighbouring cells B are mutually affected.

Additionally, the thermal stresses in the cell A and the neighbouring cells B are mutually affected. In contrast to [1]–[13], [15]–[22], this effect is explicitly determnined [14].

Figure 1.3 shows the ellipse E with the dimensiions a, b along the axes x, y, respectively. The ellipse E is described by the function

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1. \tag{1.2}$$

Any point P of the ellipse E is described by the coordinates [23]

$$x = a\cos\alpha, \quad y = b\sin\alpha, \quad \alpha \in \langle 0, 2\pi \rangle, \tag{1.3}$$

where the normal n of the ellipse E at the point P is derived [23]

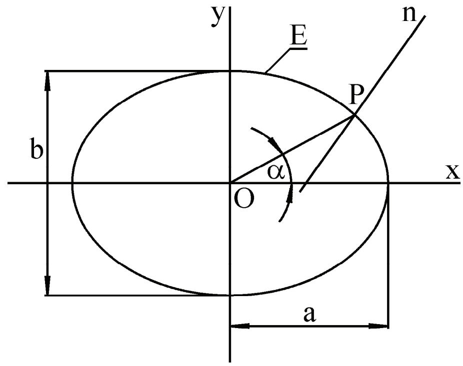


Figure 1.3: The ellipse E with the dimensions a, b along the axes x, y of the Cartesian system (Oxy), respectively, and the point P related to the angle α.

$$y = \frac{x\,a\tan\alpha}{b} - \frac{\left(a^2-b^2\right)\sin\alpha}{b}. \tag{1.4}$$

The thermal stresses are determined by the spherical coordinates $(r, \varphi\nu)$ (see Figure 1.4). The model system in Figures (1.1), (1.2) is symmetric, and then the thermal stresses are determined within the intervals $\varphi \in \langle 0, \pi/2\rangle$, $\nu \in \langle 0, \pi/2\rangle$ [1]–[22].

Figure 1.4 shows the ellipsoidal inclusion for $\varphi, \nu \in \langle 0, \pi/2\rangle$ with the centre O and with the dimensions $a_1 = O1$, $a_2 = O2$, $a_3 = O3$ along the axes x_1, x_2, x_3 of the Cartesian system (O, x_1, x_2, x_3) (see Figures (1.1), (1.2)), respectively. Finally, $\left(P, x_n, x_\varphi, x_\nu\right)$ is a Cartesian system at the point P, where the axes x_n and x_ν represents a normal and a tangent of the ellipse E_{123} at the point P, respectively, $x_{12}x_3 \perp x_1x_2$, $x_{12} \subset x_1x_2$, $x_\varphi \perp x_{12}$. Figure 1.5 shows the cross section $O567$ of the cubic cell in the plane $x_{12}x_3$ (see Figures 1.2, 1.4). The angle $\nu \in \langle 0, \pi/2\rangle$ defines a position of the point P with the Cartesian system $\left(P, x_n, x_\varphi, x_\nu\right)$ (see Figure 1.4) for $\nu = \nu_0$ (see Figure 1.5a), $\nu \in \langle 0, \nu_0)$ (see Figure 1.5b), $\nu \in (\nu_0, \pi/2\rangle$ (see Figure 1.5c). The points P_1, P_2 represent intersections of the normal x_n with $O567$.

With regard to Equations (1.2)–(1.4), the angle ν_0 represents a root of the following equation [24]

$$\frac{\cos\nu_0}{a_3}\left[\frac{d\sqrt{a_1^2\cos^2\varphi + a_2^2\sin^2\varphi}}{2f(\varphi)\sin\nu_0} + a_3^2 - \left(a_1^2\cos^2\varphi + a_2^2\sin^2\varphi\right)\right] - \frac{d}{2} = 0,$$

$$f(\varphi) = \cos\varphi, \quad \varphi \in \left\langle 0, \frac{\pi}{4}\right\rangle; \quad f(\varphi) = \sin\varphi, \quad \varphi \in \left\langle \frac{\pi}{4}, \frac{\pi}{2}\right\rangle, \tag{1.5}$$

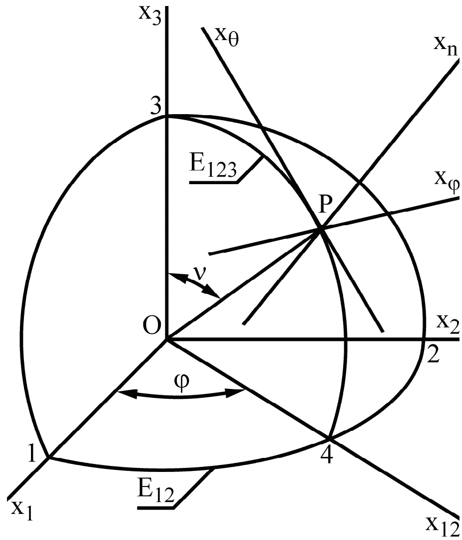


Figure 1.4: The inclusion with the centre O and with the dimensions $a_1 = O1$, $a_2 = O2$, $a_3 = O3$ along the axes x_1, x_2, x_3 of the Cartesian system (O, x_1, x_2, x_3), respectively, where E_{12}, E_{123} represent ellipses in the planes x_1x_2, $x_{12}x_3$, respectively, and $x_{12}x_3 \perp x_1x_2$, $(x_{12} \subset x_1x_2$, $x_\varphi \perp x_{12}$. The point P on the inclusion surface is defined by $\varphi, \nu \in \langle 0, \pi/2 \rangle$, $\nu \in \langle 0, \pi/2 \rangle$, and $\left(P, x_n, x_\varphi, x_\nu\right)$ is a Cartesian system at the point P, where $P \subset E_{123}$. The axes x_n and x_ν represents a normal and a tangent of the ellipse E_{123} at the point P, respectively.

and this root is determined by a numerical method. The angle $\theta = \angle(x_n, x_3)$ is derived as [24]

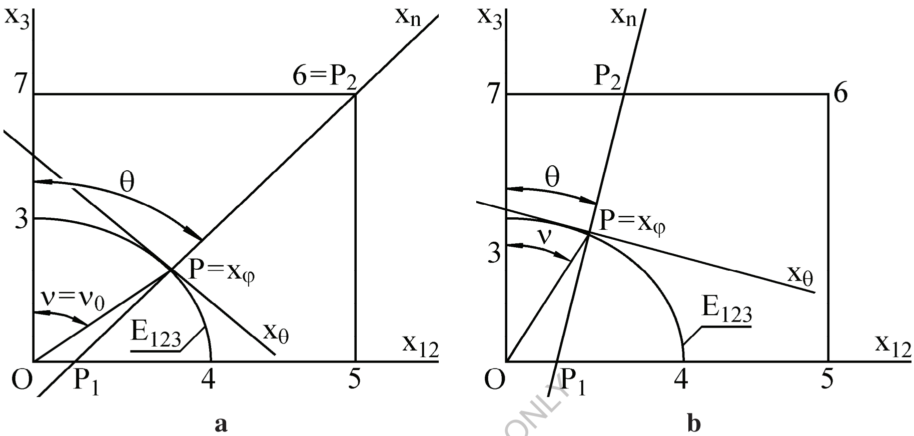
$$\cos\theta = \frac{\sqrt{a_1^2 \cos^2\varphi + a_2^2 \sin^2\varphi}}{\sqrt{a_1^2 \cos^2\varphi + a_2^2 \sin^2\varphi + (a_3 \tan\nu)^2}},$$
$$\sin\theta = \frac{a_3}{\sqrt{\left(a_1^2 \cos^2\varphi + a_2^2 \sin^2\varphi\right) \cot\nu^2 + a_3^2}}. \tag{1.6}$$

Consequently, we get [23]

$$\frac{\partial}{\partial\theta} = \left(\frac{\partial\theta}{\partial\nu}\right)^{-1} \frac{\partial}{\partial\varphi} = \Theta \frac{\partial}{\partial\nu}, \tag{1.7}$$

where the function $\Theta = \Theta(\varphi)$ has the form [24]

$$\Theta = \frac{\sqrt{a_1^2 \cos^2 \varphi + a_2^2 \sin^2 \varphi}}{a_3} \left[\frac{(a_3 \sin \nu)^2}{a_1^2 \cos^2 \varphi + a_2^2 \sin^2 \varphi} + \cos^2 \nu \right]. \qquad (1.8)$$



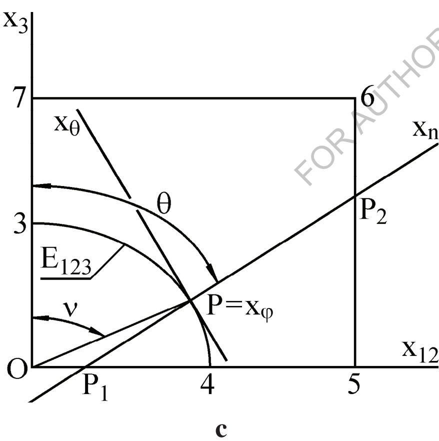


Figure 1.5: The angle $\nu \in \langle 0, \pi/2 \rangle$ defines a position of the point P with the Cartesian system $\left(P, x_n, x_\varphi, x_\nu\right)$ (see Figure 1.4) for (a) $\nu = \nu_0$, (b) $\nu \in \langle 0, \nu_0 \rangle$, (c) $\nu \in \langle \nu_0, \pi/2 \rangle$, where ν_0 is given by Equation (1.5). The points P_1, P_2 represent intersections of the normal x_n with $O567$, where $O567$ is a cross section of the cubic cell in the plane $x_{12}x_3$ (see Figures 1.2, 1.4). The angle $\theta \angle (x_n, x_3)$ is given by Equation (1.5).

As analysed in [1]-[20], due to the symmetry of the model system, any point P on the matrix-inclusion boundary exhibits the displacement u_n along x_n. Consequently, any point P of the normal x_n exhibits u_n along x_n, i.e., $u_\varphi = u_\nu = 0$ [1]-[20], where u_φ, u_ν are displacements along the axes x_φ, x_ν, respectively.

As presented in [1]–[22], the thermal stresses, which are determined along the axes x_n, x_φ, x_θ of the Cartesian system $\left(P, x_n, x_\varphi, x_\theta\right)$, represent function of the spherical coordinates (x_n, φ, θ) for $\varphi, \theta \in \langle 0, \pi/2 \rangle$. The intervals $x_n \in \langle 0, x_{IN} \rangle$ and $x_n \in$

$\langle x_{IN}, x_M \rangle$ are related to the ellipsoidal inclusion and the cell matrix, where $P = P_1$, $P \subset E_{123}$ and $P = P_2$ for $x_n = 0$, $x_n = x_{IN}$ and $x_n = x_M$ (see Figure 1.5), respectively. Finally, we get [24]

$$
\begin{aligned}
x_{IN} &= P_1 P = a_3 \sqrt{\left(\frac{a_3 \sin\nu}{a_{12}}\right)^2 + \cos^2\nu}, \\
x_M &= P P_2 = \sqrt{\left(\frac{\sin\nu}{a_{12}}\right)^2 \left(\frac{d \cos\nu}{2 a_3} - a_3^2\right)^2 + \left(\frac{a_{12} \cos\nu}{a_3}\right)^2 (x_{122} - a_{12})^2}, \\
x_{122} &= \frac{d}{2 f(\varphi) \sin\nu}, \quad a_{12} = \sqrt{a_1^2 \cos^2\varphi + a_2^2 \sin^2\varphi}.
\end{aligned}
\tag{1.9}
$$

Chapter 2

Mechanics of Elastic Solid Continuum

2.1 Fundamental Equations

As analysed in [1]-[20], any point P of the normal x_n exhibits the displacement u_n along x_n. The thermal stresses are determined along the axes x_n, x_φ, x_θ of the Cartesian system $(P, x_n, x_\varphi, x_\theta)$. Fundamental equations of mechanics of a solid continuum are represented by Cauchy's equations, the equilibrium equations and Hooke's law. Cauchy's equations represent functions of strains and displacements. With respect to the normal displacement u_n, Cauchy's equations have the forms [1]-[20, 22]

$$\varepsilon_n = \frac{\partial u_n}{\partial x_n}, \tag{2.1}$$

$$\varepsilon_\varphi = \varepsilon_\theta = \frac{u_n}{x_n}, \tag{2.2}$$

$$\varepsilon_{n\varphi} = \varepsilon_{\varphi n} = \frac{1}{x_n}\frac{\partial u_n}{\partial \varphi}, \tag{2.3}$$

$$\varepsilon_{n\theta} = \varepsilon_{\theta n} = \frac{\Theta}{x_n}\frac{\partial u_n}{\partial \nu}, \tag{2.4}$$

where ε_n is a normal strain along the axis x_n, and Θ is given by Equation (1.8). Consequently, ε_φ and ε_θ are tangential strains along the axes x_φ and x_θ, respectively. Finally, $\varepsilon_{n\varphi}$, $\varepsilon_{n\theta}$ and $\varepsilon_{\varphi n}$, $\varepsilon_{\theta n}$ represent shear strains along the axes x_n and x_φ, x_θ, respectively. Due to $u_\varphi = u_\nu = 0$, we get $\varepsilon_{\varphi\nu} = \varepsilon_{\nu\varphi} = 0$ [1]–[22], where u_φ, u_ν are displacements along the axes x_φ, x_ν, respectively, and $\varepsilon_{\varphi\nu}$ is a shear strain. As presented in [1]–[22], the equilibrium equations are derived as

$$2\sigma_n - \sigma_\varphi - \sigma_\nu + x_n \frac{\partial \sigma_n}{\partial x_n} + \frac{\partial \sigma_{n\varphi}}{\partial \varphi} + \Theta \frac{\partial \sigma_{n\theta}}{\partial \nu} = 0, \tag{2.5}$$

$$\frac{\partial \sigma_\varphi}{\partial \varphi} + 3\sigma_{n\varphi} + x_n \frac{\partial \sigma_{n\varphi}}{\partial x_n} = 0, \tag{2.6}$$

$$\Theta \frac{\partial \sigma_{\theta}}{\partial \nu} + 3\sigma_{n\theta} + x_n \frac{\partial \sigma_{n\theta}}{\partial x_n} = 0, \tag{2.7}$$

where σ_n is a normal stess along the axis x_n. Consequently, σ_φ and σ_θ are tangential stresses along the axes x_φ and x_θ, respectively. Finally, $\sigma_{n\varphi}$, $\sigma_{n\theta}$ and $\sigma_{\varphi n}$, $\sigma_{\theta n}$ represent shear stresses along the axes x_n and x_φ, x_θ, respectively, where $\sigma_{n\varphi} = \sigma_{\varphi n}$, $\sigma_{n\theta} = \sigma_{\theta n}$. Due to $\varepsilon_{\varphi\nu} = \varepsilon_{\nu\varphi} = 0$, we get $\sigma_{\varphi\nu} = \sigma_{\nu\varphi} = 0$ [1]–[22], where $\sigma_{\varphi\nu}$ is a shear stress. With regard to $\varepsilon_{\varphi\theta} = 0$, $\sigma_{\varphi\theta} = 0$, Hooke's law has the form [1]–[20, 22]

$$\varepsilon_n = s_{11}\sigma_n + s_{12}\left(\sigma_\varphi + \sigma_\theta\right), \tag{2.8}$$

$$\varepsilon_\varphi = s_{12}\left(\sigma_n + \sigma_\theta\right) + s_{11}\sigma_\varphi, \tag{2.9}$$

$$\varepsilon_\theta = s_{12}\left(\sigma_n + \sigma_\varphi\right) + s_{11}\sigma_\theta, \tag{2.10}$$

$$\varepsilon_{n\theta} = s_{44}\sigma_{n\theta}, \tag{2.11}$$

$$\varepsilon_{n\varphi} = s_{44}\sigma_{n\varphi}, \tag{2.12}$$

where s_{11}, s_{12}, s_{44} are derived as [25]

$$s_{11} = \frac{1}{E}, \quad s_{12} = -\frac{\mu}{E}, \quad s_{44} = \frac{2\left(1+\mu\right)}{E}. \tag{2.13}$$

Finally, E and μ are Young's modulus and Poisson's ratio, respectively. In case of the ellipsoidal inclusion and the cell matrix, we get $E = E_{IN}$, $\mu = \mu_{IN}$ and $E = E_M$, $\mu = \mu_M$, respectively. With regard to Equations (2.1)–(2.4), (2.8)–(2.12), we get [1]–[22]

$$\sigma_n = \left(c_1 + c_2\right) \frac{\partial u_n}{\partial x_n} - 2\,c_2\,\frac{u_n}{x_n}, \tag{2.14}$$

$$\sigma_\varphi = \sigma_\theta = -c_2\,\frac{\partial u_n}{\partial x_n} + c_1 \frac{u_n}{x_n}, \tag{2.15}$$

$$\sigma_{n\varphi} = \frac{1}{s_{44}x_n}\,\frac{\partial u_n}{\partial \varphi}, \tag{2.16}$$

$$\sigma_{n\theta} = \frac{\Theta}{s_{44}x_n}\,\frac{\partial u_n}{\partial \nu}, \tag{2.17}$$

where c_1, c_2, c_3 (see Equation (2.24)) have the forms

$$c_1 = \frac{E}{(1+\mu)(1-2\mu)}, \quad c_2 = -\frac{\mu E}{(1+\mu)(1-2\mu)}, \quad c_3 = -4(1-\mu) < 0, \qquad (2.18)$$

and $c_3 < 0$ due to $\mu < 0.5$ for real isotropic components [26]–[32].

Let $a_{1i} = \cos[\angle(x_1, x_i)]$ $(i = n, \varphi, \theta)$ represent a direction cosine of an angle formed by the axes x_1, x_i (see Figures 1.4, 1.5). With regard to Figures 1.4, 1.5, the coefficient $a_{1i} = \cos[\angle(x_1, x_i)]$ $(i = n, \varphi, \theta)$ is derived as

$$\begin{aligned} &a_{1n} = \cos\varphi\sin\theta, \quad a_{1\varphi} = \sin\varphi\sin\theta, \quad a_{1\theta} = \cos\theta, \\ &a_{\varphi 1} = -\sin\varphi, \quad a_{\theta 1} = -\cos\varphi\cos\theta, \end{aligned} \qquad (2.19)$$

where $\cos\theta$, $\sin\theta$ are given by Equation (1.6). The stress σ_1 along the axis x_1 has the form

$$\sigma_1 = a_{1n}\sigma_n + a_{1\varphi}\sigma_\varphi + a_{1\theta}\sigma_\theta + a_{1n}\left(\sigma_{n\varphi} + \sigma_{n\theta}\right) + a_{1\varphi}\sigma_{\varphi n} + a_{1\theta}\sigma_{\theta n}. \qquad (2.20)$$

With regard to Equations (2.14)-(2.17) and due to $\sigma_{n\varphi} = \sigma_{\varphi n}$, $\sigma_{n\theta} = \sigma_{\theta n}$ [25], we get

$$\sigma_1 = \gamma_1 \frac{\partial u_n}{\partial x_n} + \gamma_2 \frac{u_n}{x_n} + \frac{1}{s_{44} x_n}\left(\gamma_3 \frac{\partial u_n}{\partial \varphi} + \gamma_4 \frac{\partial u_n}{\partial \nu}\right), \qquad (2.21)$$

where γ_i $(i = 1, \ldots, 4)$ is derived as

$$\begin{aligned} &\gamma_1 = a_{1n}(c_1 + c_2) - \left(a_{1\varphi} + a_{1\theta}\right)c_2, \quad \gamma_2 = \left(a_{1\varphi} + a_{1\theta}\right)c_1 - 2a_{1n}c_2, \\ &\gamma_3 = a_{1n} + a_{1\varphi}, \quad \gamma_4 = \Theta\left(a_{1n} + a_{1\theta}\right), \end{aligned} \qquad (2.22)$$

and Θ is given by Equation (1.8). As presented in Chapter 8, the analytical models of the micro-strengthening $\sigma_{st} = \sigma_{st}(x_1)$ and the macro-strengthening $\overline{\sigma_{st}}$ result from the stress σ_1 (see Equations (2.21), (2.22)).

Let Equations (2.14)–(2.17) be substituted to Equation (2.18) and to $[\partial \text{Eq.}(2.6)/\partial\varphi]$ + $\Theta\,[\partial \text{Eq.}(2.7)/\partial\nu]$. Consequently, Equations (2.5)–(2.7) are derived as

$$x_n^2 \frac{\partial^2 u_n}{\partial x_n^2} + 2x_n \frac{\partial u_n}{\partial x_n} - 2u_n + \frac{U_n}{s_{44}(c_1 + c_2)} = 0, \qquad (2.23)$$

$$x_n \frac{\partial U_n}{\partial x_n} = c_3 U_n, \qquad (2.24)$$

where U_n is derived as

$$U_n = \frac{\partial^2 u_n}{\partial \varphi^2} + \Theta^2 \frac{\partial^2 u_n}{\partial \nu^2}. \tag{2.25}$$

The system of the differential equations (2.23), (2.25) is solved by the mathematical procedures in Sections 3.1, 4.1, 5.1, 6.1, 7.1.

2.2 Elastic Energy

As analysed in [1]–[22] with respect to the different mathematical procedures (see Sections 3.1, 4.1, 5.1, 6.1, 7.1), such a mathematical solution, which exhibits a minimum value of the elastic energy W_C of the cubic cell, is considered, where W_{IN} and W_M is elastic energy, which is accumulated in the volume V_{IN} and V_M of the ellipsoidal inclusion and the cell matrix, respectively. The elastic energy density w is derived as [25]

$$w = \frac{1}{2}\left(\varepsilon_n \sigma_n + \varepsilon_\varphi \sigma_\varphi + \varepsilon_\theta \sigma_\theta\right) + \varepsilon_{n\varphi}\sigma_{n\varphi} + \varepsilon_{n\theta}\sigma_{n\theta}, \tag{2.26}$$

and W_{IN}, W_M and W_C have the forms

$$\begin{aligned} W_{IN} &= \int\limits_{V_{IN}} w_{IN}\, dV_{IN} = 8 \int\limits_0^{\pi/2} \int\limits_0^{\pi/2} \int\limits_0^{x_{IN}} w_{IN}\, x_n^2\, dx_n\, d\varphi\, d\nu, \\ W_M &= \int\limits_{V_M} w_M\, dV_M = 8 \int\limits_0^{\pi/2} \int\limits_0^{\pi/2} \int\limits_{x_{IN}}^{x_M} w_M\, x_n^2\, dx_n\, d\varphi\, d\nu, \\ W_C &= W_{IN} + W_M. \end{aligned} \tag{2.27}$$

2.3 Mathematical Boundary Conditions

The mathematical solutions of the system of the differential equations (2.23), (2.25) include integration constants. As presented in [1]–[22], these constants are determined, using Cramer's rule (see Chapter 8) [23], by the following mathematical boundary conditions for the ellipsoidal inclusion and the cell matrix.

2.3.1 Cell Matrix. Iteration Method

The iteration method is performed by the following mathematical procedure. In case of the first iteration, i.e., $N = 1$, the mathematical boundary conditions for the cell matrix are derived as [1]–[22]

$$\left[\sigma_{nM}^{(1)}\right]_{x_n=x_{IN}} = -p_n^{(1)}, \tag{2.28}$$

$$\left[u_{nM}^{(1)}\right]_{x_n=x_M} = 0. \tag{2.29}$$

With regard to $\left[\varepsilon_{\varphi M}^{(1)}\right]_{x_n=x_M} = -p_n^{(1)}\rho_M$, $\left[\varepsilon_{\varphi IN}^{(1)}\right]_{x_n=x_{IN}} = -p_n^{(1)}\rho_{IN}$ [1]–[22], the normal stress $p_n^{(1)}$ on the matrix-inclusion boundary, i.e., for $x_n = P_1P = x_{IN}$ (see Figure 1.5), which acts along the axis x_n (see Figures (1.4), (1.5)), has the form [1]–[22]

$$p_n^{(1)} = \frac{(\alpha_{IN} - \alpha_M)(T_r - T)}{\rho_M + \rho_{IN}}, \tag{2.30}$$

where $T_r = (0.35 - 0.4) \times T_m$ [26] and T_m is relaxation and melting temperature of a real composite system, respectively, T is final temperature of a cooling process. The coefficients ρ_M and ρ_{IN} are given by Equations (3.11), (4.11), (4.14), (4.17), (5.10), (5.13), (5.16), (6.9), (6.12), (6.15), (7.10), (7.13), (7.16) and (3.14), (6.20), respectively, with respect to a minimum value of the elastic energy W_C (see Equation (2.27)). In case of the $(N+1)$-th iteration ($N = 1,2,3,\dots$), the mathematical boundary conditions are derived as

$$\left[\sigma_{nM}^{(N+1)}\right]_{x_n=x_{IN}} = -p_n^{(N+1)}, \quad N = 1,2,3,\dots, \tag{2.31}$$

$$\left[\sigma_{nM}^{(N+1)}\right]_{x_n=x_M} = -\sigma_{nB}^{(N)}, \quad N = 1,2,3,\dots, \tag{2.32}$$

$$\left[u_{nM}^{(N+1)}\right]_{x_n=x_M} = 0, \quad N = 1,2,3,\dots. \tag{2.33}$$

Similarly, with regard to $\left[\varepsilon_{\varphi M}^{(N+1)}\right]_{x_n=x_M} = -p_n^{(N+1)}\phi_M + \sigma_{nB}^{(N)}\phi_B$, $\left[\varepsilon_{\varphi IN}^{(1)}\right]_{x_n=x_{IN}} = -p_n^{(1)}\rho_{IN}$ [1]–[22], the normal stress $p_n^{(N+1)}$ on the matrix-inclusion boundary has the form

$$p_n^{(N+1)} = \frac{(\alpha_{IN} - \alpha_M)(T_r - T) + \sigma_{nB}^{(N)}\phi_B}{\phi_M + \rho_{IN}}, \quad N = 1,2,3,\dots, \tag{2.34}$$

where ϕ_M, ϕ_B are given by Equations (4.20), (5.19), (6.18), (7.19), (7.22), (7.25).

The normal stress $\sigma_{nB} = \sigma_{nB}^{(N)}$ is determined by the following mathematical procedure. With regard to Figure 1.2, the stresses in the cells B affect those in the cell A. Let P represent a point on the cell boundary with the coordinates (x_M, φ, ν) and

$(x_M, -\varphi, \nu)$ for the cells A and B, respectively. Let $\left(Px_{nA}x_{\varphi A}x_{\theta A}\right)$ and $\left(Px_{nB}x_{\varphi B}x_{\theta B}\right)$ represent coordinate systems at P in the cells A and B (see Figure 1.4), respectively.

As presented in [14], the effect of the cell B is represented by the normal stress σ_{nB}, which acts at the point P along the axis x_{nA}. The stress σ_{nB}, which is a projection of σ_{nB}, $\sigma_{\varphi B}$, $\sigma_{\theta B}$ onto x_{nA} at P (i.e., for $x_{nB} = x_M$), is derived as [14]

$$\begin{aligned} \sigma_{nB} = {} & \vartheta_1 \left(\sigma_{nM}\right)_{x_n=x_M} + \vartheta_2 \left(\sigma_{\varphi M}\right)_{x_n=x_M} + \vartheta_3 \left(\sigma_{\theta M}\right)_{x_n=x_M} \\ & + \left(\vartheta_1 + \vartheta_2\right) \left(\sigma_{n\varphi M}\right)_{x_n=x_M} + \left(\vartheta_1 + \vartheta_3\right) \left(\sigma_{n\theta M}\right)_{x_n=x_M}, \end{aligned} \tag{2.35}$$

where σ_{nM}, $\sigma_{\varphi M}$, $\sigma_{\theta M}$, $\sigma_{n\varphi M}$, $\sigma_{n\theta M}$ are determined by the mathematical boundary conditions (2.28), (2.29) (see Equations (2.14)–(2.17)), and ϑ_i $(i = 1,2,3)$ has the form [14]

$$\begin{aligned} & \vartheta_1 = 1 - 2\sin^2\varphi \sin^2\nu, \quad \vartheta_2 = \sqrt{2}\sin\left(\frac{\pi}{4} - \varphi\right)\sin\varphi \sin\nu, \\ & \vartheta_3 = \frac{1}{2}\sin\left(2\varphi - \frac{\pi}{2}\right)\sin 2\nu. \end{aligned} \tag{2.36}$$

2.3.2 Ellipsoidal Inclusion

In case of the ellipsoidal inclusion we get [1]–[22]

$$\left(u_n\right)_{x_n=0} = 0, \tag{2.37}$$

$$\left(\sigma_{nIN}\right)_{x_n=x_{IN}} = -p_n^{(N)}, \tag{2.38}$$

where x_{IN} is given by Equation (1.9). Additionally, the conditions $\left(u_{nIN}\right)_{x_n\to 0} \not\longrightarrow \pm\infty$, $\left(\varepsilon_{IN}\right)_{x_n\longrightarrow 0} \not\longrightarrow \pm\infty$, $\left(\sigma_{IN}\right)_{r\to 0} \not\longrightarrow \pm\infty$ are required to be fulfilled [1]–[22]. The normal stress $p_n^{(N)}$ (see Equations (2.30), (2.34)), which acts along the axis x_n (see Figure 1.4) on the matrix-inclusion boundary, is related to the N-th iteration (see Section 2.3.1), where $N = 1,2,3,\ldots$.

2.4 Energy Analysis

The normal stress $\sigma_{nB} = \sigma_{nB}^{(N)}$, which is a function of the radial stress $p_n = p_n^{(N)}$, is derived by 13 mathematical solutions (see Equations (3.9), (4.9), (4.12), (4.15), (5.8), (5.11), (5.14), (6.7), (6.10), (6.13), (7.8), (7.11), (7.14)). If $N = 1$ or $N + 1 = 2,3,4,\ldots$, then $p_n = p_n^{(1)}$ or $p_n = p_n^{(N+1)}$ in Equations (3.9), (4.9), (4.12), (4.15), (5.8), (5.11), (5.14), (6.7), (6.10), (6.13), (7.8), (7.11), (7.14) is given by Equation (2.30) or (2.34), respectively. Consequently, the mathematical boundary conditions (2.31)–(2.33) for the cell matrix result in 6 mathematical solutions, which are given by Equations (4.18), (5.17), (6.16), (7.17), (7.20), (7.23). Finally, two mathematical solutions for the ellipsoidal inclusion are given by Equations (3.12), (6.19).

As analysed in [1]–[15], W_C represent the total potential energy W_T [25], i.e., $W_C = W_T$. Consequently, with respect to 156 mathematical solutions, i.e., $13 \times 6 \times 2$, such a combination of the mathematical solutions is considered to result in a minimum value of the elastic energy W_C (see Equation (2.27)) [25]. Additionally, $W_C = W_C(N)$ is assumed to represent a decreasing discrete function of $N = 1, 2, 3, \dots$ with a minimum value for $N = N_{max} \subset \{1, 2, 3, \dots\}$ or $N \to \infty$. If not, then we get $N_{max} = 2$.

Chapter 3

Mathematical Model 1

3.1 Mathematical Procedure

Let the mathematical procedure $x_n\left[\partial \text{Eq.}(2.24)/\partial x_n\right]$ be performed, and then we get [1]–[22]

$$x_n^2\,\frac{\partial^2 U_n}{\partial x_n^2}+(1-c_3)\,x_n\,\frac{\partial U_n}{\partial x_n}=0, \tag{3.1}$$

where $c_3<0$ and $U_n=U_n(x_n,\varphi,\theta)$ are given by Equations (2.18) and (2.25), respectively. Let Equation (2.24) be substituted to Equation (3.1), and then we get [1]–[22]

$$x_n^2\,\frac{\partial^2 U_n}{\partial x_n^2}+c_3\,(1-c_3)\,U_n=0. \tag{3.2}$$

Let U_n be assumed in the form $U_n=x_n^{\lambda}$, then we get [1]–[22]

$$U_n=C_1\,x_n^{\lambda_1}+C_2\,x_n^{\lambda_2}, \tag{3.3}$$

where C_1, C_2 are integration constants, which are determined by the mathematical boundary conditions in Section 2.3, and λ_1, λ_2, with respect to $\mu<0.5$ for a real isotropic material [26], have the forms [1]–[22]

$$\lambda_1=\frac{1}{2}\left[1+\sqrt{1+16\,(1-\mu)\,[1+4\,(1-\mu)]}\right]>3,$$
$$\lambda_2=\frac{1}{2}\left[1-\sqrt{1+16\,(1-\mu)\,[1+4\,(1-\mu)]}\right]<-2. \tag{3.4}$$

Let Equation (3.3) be substituted to Equation (2.23), and then we get [1]–[22]

$$x_n^2\,\frac{\partial^2 u_n}{\partial x_n^2}+2\,x_n\,\frac{\partial u_n}{\partial x_n}-2\,u_n=C_1\,x_n^{\lambda_1}+C_2\,x_n^{\lambda_2}. \tag{3.5}$$

The mathematical solution of Equation (3.5), which is determined by Wronskian's method [23], is derived as

$$u_n = C_1 x_n^{\lambda_1} + C_2 x_n^{\lambda_2}. \tag{3.6}$$

With regard to Equations (2.1)–(2.4), (2.14)–(2.17), (2.21), (2.26), (3.6), we get

$$\begin{aligned}
\varepsilon_n &= C_1 \lambda_1 x_n^{\lambda_1 - 1} + C_2 \lambda_2 x_n^{\lambda_2 - 1}, \\
\varepsilon_\varphi &= C_1 x_n^{\lambda_1 - 1} + C_2 x_n^{\lambda_2 - 1}, \\
\varepsilon_{n\varphi} &= s_{44} \sigma_{n\varphi} = \frac{\partial C_1}{\partial \varphi} x_n^{\lambda_1 - 1} + \frac{\partial C_2}{\partial \varphi} x_n^{\lambda_2 - 1}, \\
\varepsilon_{n\theta} &= s_{44} \sigma_{n\theta} = \Theta \left(\frac{\partial C_1}{\partial \nu} x_n^{\lambda_1 - 1} + \frac{\partial C_2}{\partial \nu} x_n^{\lambda_2 - 1} \right), \\
\sigma_n &= C_1 \xi_1 x_n^{\lambda_1 - 1} + C_2 \xi_2 x_n^{\lambda_2 - 1}, \\
\sigma_\varphi &= \sigma_\theta = C_1 \xi_3 x_n^{\lambda_1 - 1} + C_2 \xi_4 x_n^{\lambda_2 - 1},
\end{aligned} \tag{3.7}$$

where Θ, s_{44} is given by Equations (1.8), (2.13), respectively. The coefficients ξ_i, ξ_{2+i}, ξ_5, ($i = 1,2$) are derived as

$$\xi_i = \frac{E \left[\lambda_i (1-\mu) + 2\mu \right]}{(1+\mu)(1-2\mu)}, \quad \xi_{2+i} = \frac{E (1 + \lambda_i \mu)}{(1+\mu)(1-2\mu)}, \quad i = 1, 2, \tag{3.8}$$

where γ_i ($i = 1, \ldots, 4$) is given by Equation (2.22).

3.2 Cell Matrix

Due to two integration constants, i.e., C_{1M}, C_{2M} in Equation (3.6), and two mathematical boundary conditions (see Equations (2.28), (2.29)), the mathematical model in Section 3.1 is suitable to determine the stress σ_{nB}. With regard to Equations (2.28), (2.29), (2.35), (3.6), (3.7), we get

$$\begin{aligned}
\varepsilon_{nM} &= -p_n \left[\frac{\lambda_{1M}}{\zeta_{1M}} \left(\frac{x_n}{x_M} \right)^{\lambda_{1M} - 1} + \frac{\lambda_{2M}}{\zeta_{2M}} \left(\frac{x_n}{x_M} \right)^{\lambda_{2M} - 1} \right], \\
\varepsilon_{\varphi M} &= \varepsilon_{\theta M} = -p_n \left[\frac{1}{\zeta_{1M}} \left(\frac{x_n}{x_M} \right)^{\lambda_{1M} - 1} + \frac{1}{\zeta_{2M}} \left(\frac{x_n}{x_M} \right)^{\lambda_{2M} - 1} \right], \\
\varepsilon_{n\varphi M} &= s_{44M} \sigma_{n\varphi M} = -x_n^{\lambda_{1M} - 1} \frac{\partial}{\partial \varphi} \left(\frac{p_n}{\zeta_{1M} x_M^{\lambda_{1M} - 1}} \right) - x_n^{\lambda_{2M} - 1} \frac{\partial}{\partial \varphi} \left(\frac{p_n}{\zeta_{2M} x_M^{\lambda_{2M} - 1}} \right), \\
\varepsilon_{n\theta M} &= s_{44M} \sigma_{n\theta M} = -\Theta \left[x_n^{\lambda_{1M} - 1} \frac{\partial}{\partial \nu} \left(\frac{p_n}{\zeta_{1M} x_M^{\lambda_{1M} - 1}} \right) + x_n^{\lambda_{2M} - 1} \frac{\partial}{\partial \nu} \left(\frac{p_n}{\zeta_{2M} x_M^{\lambda_{2M} - 1}} \right) \right],
\end{aligned}$$

$$
\begin{aligned}
\sigma_{nM} &= -p_n \left[\frac{\xi_{1M}}{\zeta_{1M}} \left(\frac{x_n}{x_M} \right)^{\lambda_{1M}-1} + \frac{\xi_{2M}}{\zeta_{2M}} \left(\frac{x_n}{x_M} \right)^{\lambda_{2M}-1} \right], \\
\sigma_{\varphi M} = \sigma_{\theta M} &= -p_n \left[\frac{\xi_{3M}}{\zeta_{1M}} \left(\frac{x_n}{x_M} \right)^{\lambda_{1M}-1} + \frac{\xi_{4M}}{\zeta_{2M}} \left(\frac{x_n}{x_M} \right)^{\lambda_{2M}-1} \right], \\
\sigma_{nB} &= -\left[\rho_B \, p_n + \frac{\rho_B^{(\varphi)} + \rho_B^{(\nu)}}{s_{44M}} \right],
\end{aligned}
\tag{3.9}
$$

where Θ, x_{IN}, x_M, s_{44M}, λ_{iM}, ξ_{jM} ($i=1,2;\ j=1,\ldots,4$) are given by Equations (1.8), (1.9), (2.13), (3.4), (3.8), respectively. The coefficients ζ_{iM} ($i=1,2$), ρ_B, $\rho_B^{(\tau)}$ ($\tau = \varphi, \nu$) have the forms

$$
\begin{aligned}
\zeta_{iM} &= \xi_{iM} \left(\frac{x_{IN}}{x_M} \right)^{\lambda_{iM}-1} - \xi_{3-iM} \left(\frac{x_{IN}}{x_M} \right)^{\lambda_{3-iM}-1}, \\
\rho_B &= \sum_{i=1}^{2} \frac{\vartheta_1 \xi_{iM} + (\vartheta_2 + \vartheta_3)\, \xi_{2+iM}}{\zeta_{iM}}, \\
\rho_B^{(\varphi)} &= (\vartheta_1 + \vartheta_2) \left[\frac{\partial}{\partial \varphi} \left(\frac{p_n}{\zeta_{1M} x_M^{\lambda_{1M}-1}} \right) + \frac{\partial}{\partial \varphi} \left(\frac{p_n}{\zeta_{2M} x_M^{\lambda_{2M}-1}} \right) \right] \\
\rho_B^{(\nu)} &= \Theta \, (\vartheta_1 + \vartheta_3) \left[\frac{\partial}{\partial \nu} \left(\frac{p_n}{\zeta_{1M} x_M^{\lambda_{1M}-1}} \right) + \frac{\partial}{\partial \nu} \left(\frac{p_n}{\zeta_{2M} x_M^{\lambda_{2M}-1}} \right) \right], \quad i = 1,2,
\end{aligned}
\tag{3.10}
$$

where γ_{iM}, ϑ_j ($i=1,\ldots,4;\ j=1,2,3$) are given by Equations (2.22), (2.32), respectively. The normal stress p_n is given by Equations (2.30) or (2.34) for $N = 1$ or $N+1 = 1,2,\ldots,$ respectively. With regard to Equation (3.9) and $\left[\varepsilon_{\varphi M}^{(1)} \right]_{x_n = x_M} = -p_n^{(1)} \rho_M$ [1]–[22], the coefficient ρ_M in Equation (2.30) for $N = 1$ is derived as

$$
\rho_M = \frac{1}{\zeta_{1M}} \left(\frac{x_{IN}}{x_M} \right)^{\lambda_{1M}-1} + \frac{1}{\zeta_{2M}} \left(\frac{x_{IN}}{x_M} \right)^{\lambda_{2M}-1}. \tag{3.11}
$$

3.3 Ellipsoidal Inclusion

In case of the ellipsoidal inclusion, we get $C_{2IN} = 0$, otherwise we get $(u_{nIN})_{x_n \to 0} \longrightarrow \pm\,\infty$, $(\varepsilon_{IN})_{x_n \longrightarrow 0} \longrightarrow \pm\,\infty$, $(\sigma_{IN})_{r \to 0} \longrightarrow \pm\infty$ due to $\lambda_2 < -2$ (see Equations (3.4), (3.6)–(3.12)). With regard to Equations (2.37), (2.38), (3.6), (3.7), we get [1]–[22]

$$
\varepsilon_{nIN} = -\,p_n \rho_{IN} \lambda_{1IN} \left(\frac{x_n}{x_{IN}} \right)^{\lambda_{1IN}-1}, \quad \varepsilon_{\varphi IN} = \varepsilon_{\theta IN} = -\,p_n \rho_{IN} \left(\frac{x_n}{x_{IN}} \right)^{\lambda_{1IN}-1},
$$

$$
\begin{aligned}
&\varepsilon_{n\varphi IN} = s_{44IN}\,\sigma_{n\varphi IN} = -x_n^{\lambda_{1IN}-1}\,\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\rho_{IN}}{x_{IN}^{\lambda_{1IN}-1}}\right),\\
&\varepsilon_{n\nu IN} = s_{44IN}\,\sigma_{r\nu IN} = -\Theta\,x_n^{\lambda_{1IN}-1}\,\frac{\partial}{\partial\nu}\left(\frac{p_n\,\rho_{IN}}{x_{IN}^{\lambda_{1IN}-1}}\right),\\
&\sigma_{nIN} = -p_n\left(\frac{x_n}{x_{IN}}\right)^{\lambda_{1IN}-1}, \quad \sigma_{\varphi IN} = \sigma_{\nu IN} = -\,p_n\,\rho_{IN}\,\xi_{3IN}\left(\frac{x_n}{x_{IN}}\right)^{\lambda_{1IN}-1},\\
&\sigma_{1IN} = \eta_{IN}\,x_n^{\lambda_{1IN}-1},\\
&w_{IN} = \kappa_{IN}\,x_n^{2(\lambda_{1IN}-1)}, \quad W_{IN} = \frac{4}{2\lambda_{1IN}+1}\int_0^{\pi/2}\int_0^{\pi/2}\kappa_{IN}\,x_{IN}^{2\lambda_{1IN}+1}\,d\varphi\,d\nu,
\end{aligned} \tag{3.12}
$$

where Θ; x_{IN}; s_{44IN}; λ_{1IN}; ξ_{1IN}, ξ_{3IN} are given by Equations (1.8); (1.9); (2.13); (3.4); (3.8); respectively. The coefficients η_{IN}, κ_{IN}, ξ_{IN} have the forms

$$
\begin{aligned}
\eta_{IN} = &-\frac{p_n\left(\lambda_{1IN}\gamma_{1IN}+\gamma_{2IN}\right)}{\xi_{1IN}\,x_n^{\lambda_{1IN}-1}} - \frac{\gamma_{3IN}}{s_{44IN}}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\xi_{1IN}\,x_n^{\lambda_{1IN}-1}}\right)\\
&-\frac{\gamma_{4IN}}{s_{44IN}}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\xi_{1IN}\,x_n^{\lambda_{1IN}-1}}\right),\\
\kappa_{IN} = &\,\xi_{IN}\left(\frac{p_n}{\xi_{1IN}\,x_{IN}^{\lambda_{1IN}-1}}\right)^2 + \frac{1}{s_{44IN}}\left[\frac{\partial C_1}{\partial\varphi}\left(\frac{p_n}{\xi_{1IN}\,x_{IN}^{\lambda_{1IN}-1}}\right)\right]^2\\
&+\frac{\Theta^2}{s_{44IN}}\left[\frac{\partial C_1}{\partial\nu}\left(\frac{p_n}{\xi_{1IN}\,x_{IN}^{\lambda_{1IN}-1}}\right)\right]^2,\\
\xi_{IN} = &\,\frac{E_{IN}\left\{\lambda_{1IN}\left[\lambda_{1IN}\left(1-\mu_{IN}\right)+4\mu_{IN}\right]+2\right\}}{2\left(1+\mu_{IN}\right)\left(1-2\mu_{IN}\right)},
\end{aligned} \tag{3.13}
$$

where γ_{iIN} ($i = 1,\ldots,4$) is given by Equation (2.22). The normal stress p_n is given by Equations (2.30) or (2.34) for $N = 1$ or $N+1 = 1,2,\ldots,$ respectively. With regard to Equation (3.12) and $\left(\varepsilon_{\varphi IN}\right)_{x_n = x_{IN}} = -p_n\,\rho_{IN}$ [1]–[22], the coefficient ρ_{IN} in Equations (2.30), (2.34) is derived as (see Equation (3.8))

$$
\rho_{IN} = \frac{1}{\xi_{1IN}} = \frac{\left(1+\mu_{IN}\right)\left(1-2\mu_{IN}\right)}{E_{IN}\left[\lambda_{1IN}\left(1-\mu_{IN}\right)+2\mu_{IN}\right]}. \tag{3.14}
$$

Chapter 4

Mathematical Model 2

4.1 Mathematical Procedure

Let the mathematical procedure $\partial^2 \text{Eq.}(2.24)/\partial x_n^2$ be performed, and then we get [1]–[22]

$$x_n \frac{\partial^3 U_n}{\partial x_n^3} + (2 - c_3) \frac{\partial^2 U_n}{\partial x_n^2} = 0, \tag{4.1}$$

where $c_3 < 0$ and $U_n = U_n(x_n, \varphi, \nu)$ are given by Equations (2.18) and (2.25), respectively. Let U_b be assumed in the form $U_n = x_n^{\lambda}$, and then we get

$$U_n = C_1 x_n + C_2 x_n^{c_3} + C_3, \tag{4.2}$$

where C_1, C_2, C_3 are integration constants, which are determined by the mathematical boundary conditions in Section 2.3. Let Equation (2.36) be substituted to Equation (2.23), and then we get

$$x_n^2 \frac{\partial^2 u_n}{\partial x_n^2} + 2x_n \frac{\partial u_n}{\partial x_n} - 2u_n = C_1 x_n + C_2 x_n^{c_3} + C_3 x_n^2. \tag{4.3}$$

The mathematical solution of Equation (4.3), which is determined by Wronskian's method (see Chapter 10) [23], is derived as

$$u_n = C_1 x_n \left(\frac{1}{3} - \ln x_n \right) + C_2 x_n^{c_3} + C_3. \tag{4.4}$$

With regard to Equations (2.1)–(2.4), (2.14)–(2.17), (2.26), (4.4), we get

$$\varepsilon_n = -C_1 \left(\frac{2}{3} + \ln x_n \right) + C_2 c_3 x_n^{c_3 - 1},$$

$$\varepsilon_\varphi = \varepsilon_\theta = C_1 \left(\frac{1}{3} - \ln x_n \right) + C_2 x_n^{c_3 - 1} + \frac{C_3}{x_n},$$

$$\varepsilon_{n\varphi} = s_{44} \sigma_{n\varphi} = \left(\frac{1}{3} - \ln x_n \right) \frac{\partial C_1}{\partial \varphi} + x_n^{c_3 - 1} \frac{\partial C_2}{\partial \varphi} + \frac{1}{x_n} \frac{\partial C_3}{\partial \varphi},$$

$$
\begin{aligned}
&\varepsilon_{n\theta} = s_{44}\,\sigma_{n\theta} = \Theta\left[\left(\frac{1}{3} - \ln x_n\right)\frac{\partial C_1}{\partial \nu} + x_n^{c_3-1}\,\frac{\partial C_2}{\partial \nu} + \frac{1}{x_n}\,\frac{\partial C_3}{\partial \nu}\right],\\
&\sigma_n = -\,C_1\left[\frac{2\,(c_1+2\,c_2)}{3} + (c_1-c_2)\ln x_n\right] + C_2\,[(c_1+c_2)\,c_3 - 2\,c_2]\,x_n^{c_3-1} - \frac{2\,C_3\,c_2}{x_n},\\
&\sigma_\varphi = \sigma_\theta = C_1\left[\frac{c_1+2\,c_2}{3} - (c_1-c_2)\ln x_n\right] + C_2\,(c_1 - c_2\,c_3)\,x_n^{c_3-1} + \frac{C_3\,c_1}{x_n},\\
&\sigma_1 = \eta_1 + \eta_2\ln x_n + \eta_3\,x_n^{c_3-1} + \frac{\eta_4}{x_n},\\
&w = C_1^2\,\kappa_1 + C_2^2\,\kappa_2 + C_3^2\,\kappa_3 + C_1\,C_2\,\kappa_4 + C_1\,C_3\,\kappa_5 + C_2\,C_3\,\kappa_6\\
&\quad + \frac{\chi_1}{s_{44}}\left[\left(\frac{\partial C_1}{\partial\varphi}\right)^2 + \Theta^2\left(\frac{\partial C_1}{\partial \nu}\right)^2\right] + \frac{\chi_2}{s_{44}}\left[\left(\frac{\partial C_2}{\partial\varphi}\right)^2 + \Theta^2\left(\frac{\partial C_2}{\partial \nu}\right)^2\right]\\
&\quad + \frac{\chi_3}{s_{44}}\left[\left(\frac{\partial C_3}{\partial\varphi}\right)^2 + \Theta^2\left(\frac{\partial C_3}{\partial \nu}\right)^2\right] + \frac{\chi_4}{s_{44}}\left(\frac{\partial C_1}{\partial\varphi}\,\frac{\partial C_2}{\partial\varphi} + \Theta^2\frac{\partial C_1}{\partial \nu}\,\frac{\partial C_2}{\partial \nu}\right)\\
&\quad + \frac{\chi_5}{s_{44}}\left(\frac{\partial C_1}{\partial\varphi}\,\frac{\partial C_3}{\partial\varphi} + \Theta^2\frac{\partial C_1}{\partial \nu}\,\frac{\partial C_3}{\partial \nu}\right) + \frac{\chi_6}{s_{44}}\left(\frac{\partial C_2}{\partial\varphi}\,\frac{\partial C_3}{\partial\varphi} + \Theta^2\frac{\partial C_2}{\partial \nu}\,\frac{\partial C_3}{\partial \nu}\right),
\end{aligned}
\qquad (4.5)
$$

where Θ, c_i $(i=1,2,3)$, s_{44} are given by Equations (1.8), (2.18), (2.13), respectively. The coefficients η_j, κ_k, χ_k $(j=1,\ldots,4;\ k=1,\ldots,6)$ are derived as

$$
\begin{aligned}
&\eta_1 = \frac{1}{3}\left[C_1\,(\gamma_2 - 2\,\gamma_1) + \frac{1}{s_{44}}\left(\gamma_3\,\frac{\partial C_1}{\partial\varphi} + \gamma_4\,\frac{\partial C_1}{\partial \nu}\right)\right],\\
&\eta_2 = -\left[C_1\,(\gamma_1+\gamma_2) + \frac{1}{s_{44}}\left(\gamma_3\,\frac{\partial C_1}{\partial\varphi} + \gamma_4\,\frac{\partial C_1}{\partial \nu}\right)\right],\\
&\eta_3 = C_2\,(\gamma_1\,c_3+\gamma_2) + \frac{1}{s_{44}}\left(\gamma_3\,\frac{\partial C_2}{\partial\varphi} + \gamma_4\,\frac{\partial C_2}{\partial \nu}\right),\\
&\eta_4 = C_3\,\gamma_2 + \frac{1}{s_{44}}\left(\gamma_3\,\frac{\partial C_3}{\partial\varphi} + \gamma_4\,\frac{\partial C_3}{\partial \nu}\right),\\
&\kappa_1 = \frac{c_2-c_1}{2}\,\ln^2 x_n + \frac{2\,(c_2-c_1)}{3}\,\ln x_n + \frac{7c_1+2\,c_2}{9},\\
&\kappa_2 = \left[\frac{c_3^2\,(c_1+c_2)}{2} + c_1\,(1-2\,c_3)\right]x_n^{2(c_3-1)}, \quad \kappa_3 = \frac{c_1}{x_n^2},\\
&\kappa_4 = c_3\,(c_1-c_2)\,x_n^{c_3-1}\ln x_n + 2\left[c_1 - \frac{c_3\,(2\,c_1+c_2)}{3}\right]x_n^{c_3-1},\\
&\kappa_5 = \frac{2\,c_1}{x_n}, \quad \kappa_6 = 0,\\
&\chi_1 = \ln^2 x_n - \frac{2}{3}\,\ln x_n + \frac{1}{9}, \quad \chi_2 = x_n^{2(c_3-1)}, \quad \chi_3 = \frac{1}{x_n^2},
\end{aligned}
$$

$$\chi_4 = \frac{2}{3} x_n^{c_3-1} - 2 x_n^{c_3-1} \ln x_n, \quad \chi_5 = \frac{2}{3x_n} - \frac{2\ln x_n}{x_n}, \quad \chi_6 = x_n^{c_3-2}, \tag{4.6}$$

where γ_i $(i=1,\ldots,4)$ is given by Equation (2.22). The integrals Φ_{iM}, Ψ_{iM} of the $\kappa_{jM} = \kappa_{jM}(x_n)$, $\chi_{jM} = \chi_{jM}(x_n)$ $(i=1,\ldots,6)$, respectively, have the forms

$$\Phi_{iM} = \int_{x_{IN}}^{x_M} \kappa_{iM} x_n^2 \, dx_n, \quad \Psi_{iM} = \int_{x_{IN}}^{x_M} \chi_{iM} x_n^2 \, dx_n, \quad i = 1,\ldots,6, \tag{4.7}$$

where x_{IN}, x_M are given by Equation (1.9), respectively. The integrals are determined by the formulae in Chapter 10 (see Equations (10.10)–(10.12)) and consequently, we get

$$\begin{aligned}
\Phi_{1M} &= \frac{c_{2M}-c_{1M}}{6}\left\{ x_M^3 \left[\left(\ln x_M - \frac{1}{3}\right)^2 + \frac{1}{9}\right] - x_{IN}^3 \left[\left(\ln x_{IN} - \frac{1}{3}\right)^2 + \frac{1}{9}\right]\right\} \\
&\quad + \frac{2\,(c_{2M}-c_{1M})}{9}\left[x_M^3 \left(\ln x_M - \frac{1}{3}\right) - x_{IN}^3 \left(\ln x_{IN} - \frac{1}{3}\right)\right] \\
&\quad + \frac{(7c_{1M} + 2\,c_{2M})\left(x_M^3 - x_{IN}^3\right)}{27}, \\
\Phi_{2M} &= \frac{1}{2\,c_{3M}+1}\left[\frac{c_{3M}^2\,(c_{1M}+c_{2M})}{2} + c_{1M}\,(1-2\,c_{3M})\right]\left(x_M^{2\,c_{3M}+1} - x_{IN}^{2\,c_{3M}+1}\right), \\
\Phi_{3M} &= c_{1M}\,(x_M - x_{IN}), \\
\Phi_{4M} &= \frac{c_{3M}\,(c_{1M}-c_{2M})}{c_{3M}+2}\left[x_M^{c_{3M}+2}\left(\ln x_M - \frac{1}{c_{3M}+2}\right) - x_{IN}^{c_{3M}+2}\left(\ln x_{IN} - \frac{1}{c_{3M}+2}\right)\right] \\
&\quad + \frac{2}{c_{3M}+2}\left[c_{1M} - \frac{c_{3M}\,(2\,c_{1M}+c_{2M})}{3}\right]\left(x_M^{c_{3M}+2} - x_{IN}^{c_{3M}+2}\right), \\
\Phi_{5M} &= c_{1M}\left(x_M^2 - x_{IN}^2\right), \quad \Phi_{6M} = 0, \\
\Psi_{1M} &= \frac{x_M^3}{3}\left[(\ln x_M - 1)\left(\ln x_M - \frac{1}{3}\right) + \frac{2}{9}\right] - \frac{x_{IN}^3}{3}\left[(\ln x_{IN} - 1)\left(\ln x_{IN} - \frac{1}{3}\right) + \frac{2}{9}\right], \\
\Psi_{2M} &= \frac{x_M^{2\,c_{3M}+1} - x_{IN}^{2\,c_{3M}+1}}{2\,c_{3M}+1}, \quad \Psi_{3M} = x_M - x_{IN}, \\
\Psi_{4M} &= \frac{2}{c_{3M}+2}\left\{ x_M^{c_{3M}+2}\left[\frac{c_{3M}+5}{3\,(c_{3M}+2)} - \ln x_M\right] - x_{IN}^{c_{3M}+2}\left[\frac{c_{3M}+5}{3\,(c_{3M}+2)} - \ln x_{IN}\right]\right\}, \\
\Psi_{5M} &= x_M^2\left(\frac{5}{6} - \ln x_M\right) - x_{IN}^2\left(\frac{5}{6} - \ln x_{IN}\right), \quad \Psi_{6M} = \frac{x_M^{c_{3M}+1} - x_{IN}^{c_{3M}+1}}{c_3+1}.
\end{aligned} \tag{4.8}$$

In case of the ellipsoidal inclusion, we get $(u_{nIN})_{x_n \to 0} \longrightarrow \pm\infty$, $(\varepsilon_{IN})_{x_n \longrightarrow 0} \longrightarrow \pm\infty$, $(\sigma_{IN})_{r \to 0} \longrightarrow \pm\infty$ due to $(\ln x_n)_{x_n \to 0} \longrightarrow \pm\infty$ and $(x_n^{c_3})_{x_n \to 0} \longrightarrow \pm\infty$ for

$c_3 < 0$ (see Equations (2.18), (4.4)). Accordingly, the mathematical solution (4.4) is suitable for the matrix.

4.2 Cell Matrix

The stress σ_{nB} is determined by two mathematical boundary conditions, i.e., by Equations (2.28), (2.29). With regard to three integration constants C_{1M}, C_{2M}, C_{3M} in Equation (4.4), the following conditions are considered to determine σ_{nB}, i.e., $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} = 0$; $C_{1M} \neq 0$, $C_{3M} \neq 0$, $C_{2M} = 0$; $C_{2M} \neq 0$, $C_{3M} \neq 0$, $C_{1M} = 0$. Consequently, the mathematical boundary conditions (2.31)–(2.33) are applied in case of $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} \neq 0$.

Conditions $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} = 0$. With regard to Equations (2.28), (2.29), (2.35), (4.4), (4.5), we get

$$\varepsilon_{nM} = \frac{p_n}{\zeta_M}\left[\frac{2}{3} + \ln x_n + c_{3M}\left(\frac{1}{3} - \ln x_M\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right],$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta_M}\left[\frac{1}{3} - \ln x_n - \left(\frac{1}{3} - \ln x_M\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right],$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = \left(\ln x_n - \frac{1}{3}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M}\right) + x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta_M x_M^{c_{3M}-1}}\left(\frac{1}{3} - \ln x_M\right)\right],$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = \Theta\left\{\left(\ln x_n - \frac{1}{3}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M}\right) + x_n^{c_{3M}-1}\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta_M x_M^{c_{3M}-1}}\left(\frac{1}{3} - \ln x_M\right)\right]\right\},$$

$$\sigma_{nM} = \frac{p_n}{\zeta_M}\left\{\frac{2\left(c_{1M} + 2c_{2M}\right)}{3} + \left(c_{1M} - c_{2M}\right)\ln x_n + \left[\left(c_{1M} + c_{2M}\right)c_{3M} - 2c_{2M}\right]\left(\frac{1}{3} - \ln x_M\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right\},$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta_M}\left[\frac{c_{1M} + 2c_{2M}}{3} - \left(c_{1M} - c_{2M}\right)\ln x_n - \left(c_{1M} - c_{2M}c_{3M}\right)\left(\frac{1}{3} - \ln x_M\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right],$$

$$\sigma_{nB} = -\frac{\rho_{1B}\,p_n}{\zeta_M} - \frac{1}{s_{44M}}\left[\rho_{1B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M}\right) + \rho_{1B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M}\right)\right]$$
$$+ \frac{\rho_{2B}\,p_n}{\zeta_M\,x_M^{c_{3M}-1}}\left(\frac{1}{3} - \ln x_M\right) + \frac{\rho_{2B}^{(\varphi)}}{s_{44M}}\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta_M\,x_M^{c_{3M}-1}}\left(\frac{1}{3} - \ln x_M\right)\right]$$
$$+ \frac{\rho_{2B}^{(\nu)}}{s_{44M}}\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta_M\,x_M^{c_{3M}-1}}\left(\frac{1}{3} - \ln x_M\right)\right], \tag{4.9}$$

where Θ, x_M, s_{44M}, c_{iM}, $(i=1,2,3)$ are given by Equations (1.8), (1.9), (2.13), (2.18), respectively. The coefficients ζ_M, ζ_{iM}, ρ_{iB}, $\rho_{iB}^{(\tau)}$ $(i=1,2;\ \tau=\varphi,\nu)$ have the forms

$$\zeta_M = \zeta_{2M} - \zeta_{1M}\left(\frac{1}{3} - \ln x_M\right), \quad \zeta_{1M} = \left[(c_{1M} + c_{2M})\,c_{3M} - 2\,c_{2M}\right]\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1},$$
$$\zeta_{2M} = -\left[\frac{2\,(c_{1M} + 2\,c_{2M})}{3} + (c_{1M} - c_{2M})\ln x_{IN}\right],$$
$$\rho_{1B} = \frac{c_{1M} + 2c_{2M}}{3}\,(\vartheta_2 + \vartheta_3 - 2\vartheta_1) + (c_{2M} - c_{1M})\,(\vartheta_1 + \vartheta_2 + \vartheta_3)\ln x_M,$$
$$\rho_{2B} = \left\{\vartheta_1\left[(c_{1M} + c_{2M})\,c_{3M} - 2\,c_{2M}\right] + (\vartheta_2 + \vartheta_3)\,(c_{1M} - c_{2M}\,c_{3M})\right\}x_M^{c_{3M}-1},$$
$$\rho_{1B}^{(\varphi)} = (\vartheta_1 + \vartheta_2)\left(\frac{1}{3} - \ln x_M\right), \quad \rho_{1B}^{(\nu)} = \Theta\,(\vartheta_1 + \vartheta_3)\left(\frac{1}{3} - \ln x_M\right),$$
$$\rho_{2B}^{(\varphi)} = (\vartheta_1 + \vartheta_2)\,x_M^{c_{3M}-1}, \quad \rho_{2B}^{(\nu)} = \Theta\,(\vartheta_1 + \vartheta_3)\,x_M^{c_{3M}-1}, \tag{4.10}$$

where ϑ_i $(i=1,2,3)$ is given by Equation (2.32). The normal stress p_n is given by Equations (2.30) or (2.34) for $N = 1$ or $N + 1 = 1, 2, \ldots,$ respectively. With regard to Equation (4.9) and $\left[\varepsilon_{\varphi M}^{(1)}\right]_{x_n = x_M} = -p_n^{(1)}\,\rho_M$ [1]–[22], the coefficient ρ_M in Equation (2.30) for $N = 1$ is derived as

$$\rho_M = \frac{1}{\zeta_M}\left[\frac{1}{3} - \ln x_{IN} - \left(\frac{1}{3} - \ln x_M\right)\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1}\right]. \tag{4.11}$$

Conditions $C_{1M} \neq 0, C_{3M} \neq 0, C_{2M} = 0$. Similarly, with regard to Equations (2.28), (2.29), (2.35), (4.4), (4.5), we get

$$\varepsilon_{nM} = \frac{p_n}{\zeta_M\,x_M}\left(\frac{2}{3} + \ln x_n\right),$$
$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta_M}\left[\frac{1}{x_M}\left(\frac{1}{3} - \ln x_n\right) - \frac{1}{x_n}\left(\frac{1}{3} - \ln x_M\right)\right],$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = -\left(\frac{1}{3} - \ln x_n\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M\, x_M}\right) + \frac{1}{x_n}\frac{\partial}{\partial\varphi}\left[\frac{p_n\,(1-3\ln x_M)}{3\,\zeta_M}\right],$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = \\ -\Theta\left\{\left(\frac{1}{3} - \ln x_n\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M\, x_M}\right) - \frac{1}{x_n}\frac{\partial}{\partial\varphi}\left[\frac{p_n\,(1-3\ln x_M)}{3\,\zeta_M}\right]\right\},$$

$$\sigma_{nM} = \frac{p_n}{\zeta_M}\left\{\frac{1}{x_M}\left[\frac{2\,(c_{1M}+2\,c_{2M})}{3} + (c_{1M}-c_{2M})\ln x_n\right] - \frac{2\,c_{2M}}{x_n}\left(\frac{1}{3} - \ln x_M\right)\right\},$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = \\ -\frac{p_n}{\zeta_M}\left\{\frac{1}{x_M}\left[\frac{c_{1M}+2\,c_{2M}}{3} - (c_{1M}-c_{2M})\ln x_n\right] - \frac{c_{1M}}{x_n}\left(\frac{1}{3} - \ln x_M\right)\right\},$$

$$\sigma_{nB} = -\frac{\rho_{1B}\,p_n}{\zeta_M x_M} - \frac{1}{s_{44M}}\left[\rho_{1B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M x_M}\right) + \rho_{1B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M x_M}\right)\right] \\ + \frac{\rho_{3B}\,p_n}{\zeta_M}\left(\frac{1}{3} - \ln x_M\right) + \frac{\rho_{3B}^{(\varphi)}}{s_{44M}}\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta_M}\left(\frac{1}{3} - \ln x_M\right)\right] \\ + \frac{\rho_{3B}^{(\nu)}}{s_{44M}}\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta_M}\left(\frac{1}{3} - \ln x_M\right)\right], \qquad (4.12)$$

where Θ, x_M, s_{44M}, c_{iM} and ρ_{1B}, $\rho_{1B}^{(\tau)}$ $(i=1,2,3;\ \tau=\varphi,\nu)$ are given by Equations (1.8), (1.9), (2.13), (2.18) and (4.10), respectively. The coefficients ζ_M, ρ_{3B}, $\rho_{3B}^{(\tau)}$ $(\tau=\varphi,\nu)$ have the forms

$$\zeta_M = \frac{1}{x_M}\left[\frac{2\,(c_{1M}+2\,c_{2M})}{3} + (c_{1M}-c_{2M})\ln x_{IN}\right] + \frac{2\,c_{2M}}{x_{IN}}\left(\frac{1}{3} - \ln x_M\right),$$

$$\rho_{3B} = \frac{1}{x_M}\left[c_{1M}\,(\vartheta_2+\vartheta_3) - 2\,c_{1M}\vartheta_1\right],$$

$$\rho_{3B}^{(\varphi)} = \frac{\vartheta_1+\vartheta_2}{x_M}, \quad \rho_{3B}^{(\nu)} = \frac{\Theta\,(\vartheta_1+\vartheta_3)}{x_M}, \qquad (4.13)$$

where ϑ_i $(i=1,2,3)$ is given by Equations (2.32). The normal stress p_n is given by Equations (2.30) or (2.34) for $N=1$ or $N+1=1,2,\ldots,$ respectively. With regard to Equation (4.12) and $\left[\varepsilon_{\varphi M}^{(1)}\right]_{x_n=x_M} = -p_n^{(1)}\rho_M$ [1]–[22], the coefficient ρ_M in Equation (2.30) for $N=1$ is derived as

$$\rho_M = \frac{1}{\zeta_M}\left[\frac{1}{x_M}\left(\frac{1}{3} - \ln x_{IN}\right) - \frac{1}{x_{IN}}\left(\frac{1}{3} - \ln x_M\right)\right]. \qquad (4.14)$$

Conditions $C_{2M} \neq 0$, $C_{3M} \neq 0$, $C_{1M} = 0$. Consequently, with regard to Equations (2.28), (2.29), (2.35), (4.4), (4.5), we get

$$
\begin{aligned}
\varepsilon_{nM} &= -\frac{p_n c_{3M}}{\zeta_M x_M}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}, \\
\varepsilon_{\varphi M} = \varepsilon_{\theta M} &= \frac{p_n}{\zeta_M x_n}\left[1-\left(\frac{x_n}{x_M}\right)^{c_{3M}}\right], \\
\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} &= -\frac{1}{x_n}\left[x_n^{c_{3M}}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M x_M^{c_{3M}}}\right)-\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M}\right)\right], \\
\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} &= -\frac{\Theta}{x_n}\left[x_n^{c_{3M}}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M x_M^{c_{3M}}}\right)-\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M}\right)\right], \\
\sigma_{nM} &= -\frac{p_n}{\zeta_M}\left\{\frac{c_{3M}(c_{1M}+c_{2M})-2c_{2M}}{x_M}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}+\frac{2c_{2M}}{x_n}\right\}, \\
\sigma_{\varphi M} = \sigma_{\theta M} &= -\frac{p_n}{\zeta_M}\left[\frac{c_{1M}-c_{2M}c_{3M}}{x_M}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}-\frac{c_{1M}}{x_n}\right], \\
\sigma_{nB} &= -\frac{\rho_{1B}\,p_n}{\zeta_M x_M^{c_{3M}}}-\frac{1}{s_{44M}}\left[\rho_{1B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M x_M^{c_{3M}}}\right)+\rho_{1B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M x_M^{c_{3M}}}\right)\right] \\
&\quad+\frac{\rho_{3B}\,p_n}{\zeta_M}+\frac{1}{s_{44M}}\left[\rho_{3B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M}\right)+\rho_{3B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M}\right)\right],
\end{aligned}
\tag{4.15}
$$

where Θ, x_M, s_{44M}, c_{iM} ($i=1,2,3$) and ρ_{jB}, $\rho_{jB}^{(\tau)}$ ($j=2,3$; $\tau=\varphi,\nu$) are given by Equations (1.8), (1.9), (2.13), (2.18) and (4.10), (4.13), respectively. The coefficient ζ_M has the form

$$
\zeta_M = \frac{1}{x_{IN}}\left\{[c_{3M}(c_{1M}+c_{2M})-2c_{2M}]\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}}+2c_{2M}\right\}. \tag{4.16}
$$

The normal stress p_n is given by Equations (2.30) or (2.34) for $N=1$ or $N+1=1,2,\ldots$, respectively. With regard to Equation (4.15) and $\left[\varepsilon_{\varphi M}^{(1)}\right]_{x_n=x_M} = -p_n^{(1)}\rho_M$ [1]–[22], the coefficient ρ_M in Equation (2.30) for $N=1$ is derived as

$$
\rho_M = \frac{1}{\zeta_M}\left[\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}}-1\right]. \tag{4.17}
$$

Conditions $C_{1M}\neq 0$, $C_{2M}\neq 0$, $C_{3M}\neq 0$. Finally, with regard to Equations (2.21), (2.27), (2.31)–(2.33), (4.4), (4.5), we get

$$
\begin{aligned}
\varepsilon_{nM} &= -\frac{1}{\zeta_M}\left[\zeta_{1M}\left(\frac{2}{3}+\ln x_n\right)-\zeta_{2M}c_{3M}x_n^{c_{3M}-1}\right], \\
\varepsilon_{\varphi M} = \varepsilon_{\theta M} &= \frac{1}{\zeta_M}\left[\zeta_{1M}\left(\frac{1}{3}-\ln x_n\right)+\zeta_{2M}x_n^{c_{3M}-1}+\frac{\zeta_{3M}}{x_n}\right],
\end{aligned}
$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = \left(\frac{1}{3} - \ln x_n\right)\frac{\partial}{\partial\varphi}\left(\frac{\zeta_{1M}}{\zeta_M}\right) + x_n^{c_3-1}\frac{\partial}{\partial\varphi}\left(\frac{\zeta_{2M}}{\zeta_M}\right) + \frac{1}{x_n}\frac{\partial}{\partial\varphi}\left(\frac{\zeta_{3M}}{\zeta_M}\right),$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = \Theta\left[\left(\frac{1}{3} - \ln x_n\right)\frac{\partial}{\partial\nu}\left(\frac{\zeta_{1M}}{\zeta_M}\right) + x_n^{c_3-1}\frac{\partial}{\partial\nu}\left(\frac{\zeta_{2M}}{\zeta_M}\right) + \frac{1}{x_n}\frac{\partial}{\partial\nu}\left(\frac{\zeta_{3M}}{\zeta_M}\right)\right],$$

$$\sigma_{nM} = -\frac{1}{\zeta_M}\left\{\zeta_{1M}\left[\frac{2\,(c_{1M} + 2\,c_{2M})}{3} + (c_{1M} - c_{2M})\ln x_n\right] - \zeta_{2M}\left[(c_{1M} + c_{2M})\,c_{3M} - 2\,c_{2M}\right]x_n^{c_{3M}-1} + \frac{2\,c_{2M}\,\zeta_{3M}}{x_n}\right\},$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = \frac{1}{\zeta_M}\left\{\zeta_{1M}\left[\frac{c_{1M} + 2\,c_{2M}}{3} - (c_{1M} - c_{2M})\ln x_n\right] + \zeta_{2M}\,(c_{1M} - c_{2M}\,c_{3M})\,x_n^{c_{3M}-1} + \frac{c_{1M}\,\zeta_{3M}}{x_n}\right\},$$

$$\sigma_{1M} = \eta_{1M} + \eta_{2M}\ln x_n + \eta_{3M}\,x_n^{c_{3M}-1} + \frac{\eta_{4M}}{x_n},$$

$$\begin{aligned} w_M = {} & \left(\frac{1}{\zeta_M}\right)^2\Big(\kappa_{1M}\,\zeta_{1M}^2 + \kappa_{2M}\,\zeta_{2M}^2 + \kappa_{3M}\,\zeta_{3M}^2 + \kappa_{4M}\,\zeta_{1M}\,\zeta_{2M} + \kappa_{5M}\,\zeta_{1M}\,\zeta_{3M} + \kappa_{6M}\,\zeta_{2M}\,\zeta_{3M}\Big) \\ & + \frac{\chi_{1M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{\zeta_{1M}}{\zeta_M}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{\zeta_{1M}}{\zeta_M}\right)\right]^2\right\} \\ & + \frac{\chi_{2M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{\zeta_{2M}}{\zeta_M}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{\zeta_{2M}}{\zeta_M}\right)\right]^2\right\} \\ & + \frac{\chi_{3M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{\zeta_{3M}}{\zeta_M}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{\zeta_{3M}}{\zeta_M}\right)\right]^2\right\} \\ & + \frac{\chi_{4M}}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{\zeta_{1M}}{\zeta_M}\right)\frac{\partial}{\partial\varphi}\left(\frac{\zeta_{2M}}{\zeta_M}\right) + \Theta^2\frac{\partial}{\partial\nu}\left(\frac{\zeta_{1M}}{\zeta_M}\right)\frac{\partial}{\partial\nu}\left(\frac{\zeta_{2M}}{\zeta_M}\right)\right] \\ & + \frac{\chi_{5M}}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{\zeta_{1M}}{\zeta_M}\right)\frac{\partial}{\partial\varphi}\left(\frac{\zeta_{3M}}{\zeta_M}\right) + \Theta^2\frac{\partial}{\partial\nu}\left(\frac{\zeta_{1M}}{\zeta_M}\right)\frac{\partial}{\partial\nu}\left(\frac{\zeta_{3M}}{\zeta_M}\right)\right] \\ & + \frac{\chi_{6M}}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{\zeta_{2M}}{\zeta_M}\right)\frac{\partial}{\partial\varphi}\left(\frac{\zeta_{3M}}{\zeta_M}\right) + \Theta^2\frac{\partial}{\partial\nu}\left(\frac{\zeta_{2M}}{\zeta_M}\right)\frac{\partial}{\partial\nu}\left(\frac{\zeta_{3M}}{\zeta_M}\right)\right], \end{aligned}$$

$$W_M = 4 \int_0^{\pi/2} \int_0^{\pi/2} \left(\frac{1}{\zeta_M} \right)^2 \Big(\Phi_{1M} \zeta_{1M}^2 + \Phi_{2M} \zeta_{2M}^2 + \Phi_{3M} \zeta_{3M}^2 + \Phi_{4M} \zeta_{1M} \zeta_{2M}$$

$$+ \Phi_{5M} \zeta_{1M} \zeta_{3M} + \Phi_{6M} \zeta_{2M} \zeta_{3M} \Big) \, d\varphi \, d\nu$$

$$+ \frac{4}{s_{44M}} \int_0^{\pi/2} \int_0^{\pi/2} \Psi_{1M} \left\{ \left[\frac{\partial}{\partial \varphi} \left(\frac{\zeta_{1M}}{\zeta_M} \right) \right]^2 + \Theta^2 \left[\frac{\partial}{\partial \nu} \left(\frac{\zeta_{1M}}{\zeta_M} \right) \right]^2 \right\} d\varphi \, d\nu$$

$$+ \frac{4}{s_{44M}} \int_0^{\pi/2} \int_0^{\pi/2} \Psi_{2M} \left\{ \left[\frac{\partial}{\partial \varphi} \left(\frac{\zeta_{2M}}{\zeta_M} \right) \right]^2 + \Theta^2 \left[\frac{\partial}{\partial \nu} \left(\frac{\zeta_{2M}}{\zeta_M} \right) \right]^2 \right\} d\varphi \, d\nu$$

$$+ \frac{4}{s_{44M}} \int_0^{\pi/2} \int_0^{\pi/2} \Psi_{3M} \left\{ \left[\frac{\partial}{\partial \varphi} \left(\frac{\zeta_{3M}}{\zeta_M} \right) \right]^2 + \Theta^2 \left[\frac{\partial}{\partial \nu} \left(\frac{\zeta_{3M}}{\zeta_M} \right) \right]^2 \right\} d\varphi \, d\nu$$

$$+ \frac{4}{s_{44M}} \int_0^{\pi/2} \int_0^{\pi/2} \Psi_{4M} \left[\frac{\partial}{\partial \varphi} \left(\frac{\zeta_{1M}}{\zeta_M} \right) \frac{\partial}{\partial \varphi} \left(\frac{\zeta_{2M}}{\zeta_M} \right) \right.$$

$$\left. + \Theta^2 \frac{\partial}{\partial \nu} \left(\frac{\zeta_{1M}}{\zeta_M} \right) \frac{\partial}{\partial \nu} \left(\frac{\zeta_{2M}}{\zeta_M} \right) \right] d\varphi \, d\nu$$

$$+ \frac{4}{s_{44M}} \int_0^{\pi/2} \int_0^{\pi/2} \Psi_{5M} \left[\frac{\partial}{\partial \varphi} \left(\frac{\zeta_{1M}}{\zeta_M} \right) \frac{\partial}{\partial \varphi} \left(\frac{\zeta_{3M}}{\zeta_M} \right) \right.$$

$$\left. + \Theta^2 \frac{\partial}{\partial \nu} \left(\frac{\zeta_{1M}}{\zeta_M} \right) \frac{\partial}{\partial \nu} \left(\frac{\zeta_{3M}}{\zeta_M} \right) \right] d\varphi \, d\nu$$

$$+ \frac{4}{s_{44M}} \int_0^{\pi/2} \int_0^{\pi/2} \Psi_{6M} \left[\frac{\partial}{\partial \varphi} \left(\frac{\zeta_{2M}}{\zeta_M} \right) \frac{\partial}{\partial \varphi} \left(\frac{\zeta_{3M}}{\zeta_M} \right) \right.$$

$$\left. + \Theta^2 \frac{\partial}{\partial \nu} \left(\frac{\zeta_{2M}}{\zeta_M} \right) \frac{\partial}{\partial \nu} \left(\frac{\zeta_{3M}}{\zeta_M} \right) \right] d\varphi \, d\nu,$$

$$\sigma_{nB} = \sum_{i=1}^{3} \frac{\rho_{iB} \zeta_{iM}}{\zeta_M} + \frac{1}{s_{44M}} \left[\rho_{iB}^{(\varphi)} \frac{\partial}{\partial \varphi} \left(\frac{\zeta_{iM}}{\zeta_M} \right) + \rho_{iB}^{(\nu)} \frac{\partial}{\partial \nu} \left(\frac{\zeta_{iM}}{\zeta_M} \right) \right], \tag{4.18}$$

where Θ, x_M, s_{44M}, c_{iM}; κ_{jM}, χ_{jM}; Φ_{jM}, Ψ_{jM}; ρ_{iB}, $\rho_{iB}^{(\tau)}$ $(i = 1,2,3;\ j = 1,\ldots,6;\ \tau = \varphi, \nu)$; are given by Equations (1.8), ((1.9), (2.13), (2.18); (4.6); (4.8); (4.11), (4.15); respectively. The coefficients ζ_{iM}, ζ_M, η_{jM} $(i = 1,2,3;\ j = 1,\ldots,4)$ have the forms

$$\zeta_{iM} = -p_n \zeta_{i1M} + \sigma_{nB} \zeta_{i2M}, \quad i = 1,2,3,$$
$$\zeta_{11M} = (c_{1M} + c_{2M}) \, c_{3M} \, x_M^{c_{3M}-1},$$

$$
\begin{aligned}
\zeta_{12M} &= [(c_{1M}+c_{2M})\,c_{3M}-2c_{2M}]\,x_{IN}^{c_{3M}-1}+\frac{2\,c_{2M}\,x_M^{c_{3M}}}{x_{IN}},\\
\zeta_{21M} &= (c_{1M}+c_{2M})\left(\frac{2}{3}+\ln x_M\right),\\
\zeta_{22M} &= \frac{2\,(c_{1M}+2\,c_{2M})}{3}+(c_{1M}-c_{2M})\ln x_{IN}-\frac{2\,c_{2M}\,x_M}{x_{IN}}\left(\frac{1}{3}-\ln x_M\right),\\
\zeta_{31M} &= -\,(c_{1M}+c_{2M})\left[\frac{2+c_{3M}}{3}+(1+c_{3M})\ln x_M\right]x_M^{c_{3M}},\\
\zeta_{32M} &= -\left[\frac{2\,(c_{1M}+2\,c_{2M})}{3}+(c_{1M}-c_{2M})\ln x_{IN}\right]x_M^{c_{3M}}\\
&\quad -[(c_{1M}+c_{2M})\,c_{3M}-2\,c_{2M}]\left(\frac{1}{3}-\ln x_M\right)x_M\,x_{IN}^{c_{3M}-1},\\
\zeta_M &= (c_{1M}+c_{2M})\\
&\quad\times\left\{-\left[\frac{2\,(c_{1M}+2\,c_{2M})}{3}+(c_{1M}-c_{2M})\ln x_{IN}\right]c_{3M}\,x_M^{c_{3M}-1}\right.\\
&\quad +[(c_{1M}+c_{2M})\,c_{3M}-2\,c_{2M}]\left(\frac{2}{3}+\ln x_M\right)x_{IN}^{c_{3M}-1}\\
&\quad \left.+\left[\frac{2+c_{3M}}{3}+(1-c_{3M})\ln x_M\right]\frac{2\,c_{2M}\,x_M^{c_{3M}}}{x_{IN}}\right\},\\
\eta_{1M} &= \frac{\zeta_{1M}\,(\gamma_{2M}-2\,\gamma_{1M})}{3\,\zeta_M}+\frac{1}{3\,s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{\zeta_{1M}}{\zeta_M}\right)+\gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{\zeta_{1M}}{\zeta_M}\right)\right],\\
\eta_{2M} &= -\,\frac{\zeta_{1M}\,(\gamma_{1M}+\gamma_{2M})}{\zeta_M}-\frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{\zeta_{1M}}{\zeta_M}\right)+\gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{\zeta_{1M}}{\zeta_M}\right)\right],\\
\eta_{3M} &= \frac{\zeta_{2M}\,(\gamma_{1M}\,c_3+\gamma_{2M})}{\zeta_M}+\frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{\zeta_{2M}}{\zeta_M}\right)+\gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{\zeta_{2M}}{\zeta_M}\right)\right],\\
\eta_{4M} &= \frac{\zeta_{3M}\gamma_{2M}}{\zeta_M}+\frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{\zeta_{3M}}{\zeta_M}\right)+\gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{\zeta_{3M}}{\zeta_M}\right)\right],
\end{aligned}
\tag{4.19}
$$

where γ_{iM} $(i=1,\ldots,4)$ is given by Equation (2.22). The normal stress p_n is given by Equations (2.30) or (2.34) for $N=1$ or $N+1=1,2,\ldots$, respectively. With regard to Equation (4.18) and $\left[\varepsilon_{\varphi M}\right]_{x_n=x_M}=-p_n\,\phi_M+\sigma_{nB}\,\phi_B$, the coefficients ϕ_M, ϕ_B in Equation (2.34) for $N+1=1,2,\ldots$ are derived as

$$
\begin{aligned}
\phi_M &= \frac{1}{\zeta_M}\left[\zeta_{11M}\left(\frac{1}{3}-\ln x_{IN}\right)+\zeta_{21M}\,x_{IN}^{c_{3M}-1}+\frac{\zeta_{31M}}{x_{IN}}\right],\\
\phi_B &= \frac{1}{\zeta_M}\left[\zeta_{12M}\left(\frac{1}{3}-\ln x_{IN}\right)+\zeta_{22M}\,x_{IN}^{c_{3M}-1}+\frac{\zeta_{32M}}{x_{IN}}\right].
\end{aligned}
\tag{4.20}
$$

Chapter 5

Mathematical Model 3

5.1 Mathematical Procedure

Let the mathematical procedure $\partial^2 \text{Eq.}(2.23)/\partial x_n^2$ be performed, and then we get [1]–[22]

$$r\frac{\partial^3 u_n}{\partial x_n^3} + 4x_n^2\frac{\partial^2 u_n}{\partial x_n^2} + \frac{x_n}{s_{44}(c_1+c_2)}\frac{\partial U_n}{\partial x_n} = 0, \tag{5.1}$$

where s_{44}, c_i $(i = 1,2,3)$ and $U_n = U_n(r,\varphi,\nu)$ are given by Equations (2.13), (2.18) and (2.25), respectively. With regard to Equations (2.24), (4.2), we get

$$x_n\frac{\partial U_n}{\partial x_n} = c_3\left(C_1 x_n + C_2 x_n^{c_3} + C_3\right), \tag{5.2}$$

where C_1, C_2, C_3 are integration constants, which are determined by the mathematical boundary conditions in Section 2.3. Let Equation (5.2) be substituted to Equation (5.1), and then we get

$$x_n^3\frac{\partial^3 u_n}{\partial x_n^3} + 4x_n^2\frac{\partial^2 u_n}{\partial x_n^2} = C_1 x_n^3 + C_2 x_n^{c_3} + C_3. \tag{5.3}$$

The mathematical solution of Equation (5.3), which is determined by Wronskian's method (see Chapter 10) [23], is derived as

$$u_n = C_1 x_n\left(\frac{4}{3} - \ln x_n\right) + C_2 x_n^{c_3} + C_3\left(\frac{1}{2} + \ln x_n\right). \tag{5.4}$$

With regard to Equations (2.1)–(2.4), (2.14)–(2.17), (2.26), (5.4), we get

$$\varepsilon_n = C_1\left(\frac{1}{3} - \ln x_n\right) + C_2 c_3 x_n^{c_3-1} + \frac{C_3}{x_n},$$
$$\varepsilon_\varphi = \varepsilon_\theta = C_1\left(\frac{4}{3} - \ln x_n\right) + C_2 x_n^{c_3-1} + \frac{C_3}{x_n}\left(\frac{1}{2} + \ln x_n\right),$$
$$\varepsilon_{n\varphi} = s_{44}\sigma_{n\varphi} = \left(\frac{4}{3} - \ln x_n\right)\frac{\partial C_1}{\partial\varphi} + x_n^{c_3-1}\frac{\partial C_2}{\partial\varphi} + \frac{1}{x_n}\left(\frac{1}{2} + \ln x_n\right)\frac{\partial C_3}{\partial\varphi},$$
$$\varepsilon_{n\theta} = s_{44}\sigma_{n\theta} = \Theta\left[\left(\frac{4}{3} - \ln x_n\right)\frac{\partial C_1}{\partial\nu} + x_n^{c_3-1}\frac{\partial C_2}{\partial\nu} + \frac{1}{x_n}\left(\frac{1}{2} + \ln x_n\right)\frac{\partial C_3}{\partial\nu}\right],$$

$$\sigma_n = C_1 \left[\frac{c_1 - 7c_2}{3} - (c_1 - c_2)\ln x_n\right] + C_2\left[(c_1 + c_2)\, c_3 - 2\,c_2\right] x_n^{c_3 - 1}$$
$$+ \frac{C_3}{x_n}(c_1 - 2\,c_2 \ln x_n),$$
$$\sigma_\varphi = \sigma_\theta = C_1 \left[\frac{4c_1 - c_2}{3} - (c_1 - c_2)\ln x_n\right] + C_2 (c_1 - c_2\, c_3)\, x_n^{c_3 - 1}$$
$$+ \frac{C_3}{x_n}\left(\frac{c_1 - 2\,c_2}{2} + c_1 \ln x_n\right),$$
$$\sigma_1 = \eta_1 + \eta_2 \ln x_n + \eta_3 x_n^{c_3 - 1} + \frac{\eta_4 + \eta_5 \ln x_n}{x_n},$$
$$w = C_1^2\,\kappa_1 + C_2^2\,\kappa_2 + C_3^2\,\kappa_3 + C_1 C_2\,\kappa_4 + C_1 C_3\,\kappa_5 + C_2 C_3\,\kappa_6$$
$$+ \frac{\chi_1}{s_{44}}\left[\left(\frac{\partial C_1}{\partial\varphi}\right)^2 + \Theta^2\left(\frac{\partial C_1}{\partial\nu}\right)^2\right] + \frac{\chi_2}{s_{44}}\left[\left(\frac{\partial C_2}{\partial\varphi}\right)^2 + \Theta^2\left(\frac{\partial C_2}{\partial\nu}\right)^2\right]$$
$$+ \frac{\chi_3}{s_{44}}\left[\left(\frac{\partial C_3}{\partial\varphi}\right)^2 + \Theta^2\left(\frac{\partial C_3}{\partial\nu}\right)^2\right] + \frac{\chi_4}{s_{44}}\left(\frac{\partial C_1}{\partial\varphi}\frac{\partial C_2}{\partial\varphi} + \Theta^2\frac{\partial C_1}{\partial\nu}\frac{\partial C_2}{\partial\nu}\right)$$
$$+ \frac{\chi_5}{s_{44}}\left(\frac{\partial C_1}{\partial\varphi}\frac{\partial C_3}{\partial\varphi} + \Theta^2\frac{\partial C_1}{\partial\nu}\frac{\partial C_3}{\partial\nu}\right) + \frac{\chi_6}{s_{44}}\left(\frac{\partial C_2}{\partial\varphi}\frac{\partial C_3}{\partial\varphi} + \Theta^2\frac{\partial C_2}{\partial\nu}\frac{\partial C_3}{\partial\nu}\right), \quad (5.5)$$

where Θ is given by Equation (1.8). The coefficients $\eta_i\ \kappa_j, \chi_j\ (i = 1, \ldots, 4;\ j = 1, \ldots, 6)$ are derived as

$$\eta_1 = \frac{1}{3}\left[C_1(\gamma_1 + 4\gamma_2) + \frac{4}{s_{44}}\left(\gamma_3\frac{\partial C_1}{\partial\varphi} + \gamma_4\frac{\partial C_1}{\partial\nu}\right)\right],$$
$$\eta_2 = -\left[C_1(\gamma_1 + \gamma_2) + \frac{1}{s_{44}}\left(\gamma_3\frac{\partial C_1}{\partial\varphi} + \gamma_4\frac{\partial C_1}{\partial\nu}\right)\right],$$
$$\eta_3 = C_2(\gamma_1 c_3 + \gamma_2) + \frac{1}{s_{44}}\left(\gamma_3\frac{\partial C_2}{\partial\varphi} + \gamma_4\frac{\partial C_2}{\partial\nu}\right),$$
$$\eta_4 = C_3\left(\gamma_1 + \frac{\gamma_2}{2}\right) + \frac{1}{2\,s_{44}}\left(\gamma_3\frac{\partial C_3}{\partial\varphi} + \gamma_4\frac{\partial C_3}{\partial\nu}\right),$$
$$\eta_5 = C_3\,\gamma_2 + \frac{1}{s_{44}}\left(\gamma_3\frac{\partial C_3}{\partial\varphi} + \gamma_4\frac{\partial C_3}{\partial\nu}\right),$$
$$\kappa_1 = \frac{c_2 - c_1}{2}\ln^2 x_n + \frac{c_1 - c_2}{3}\ln x_n + \frac{17\,c_1 + c_2}{18},$$
$$\kappa_2 = \left[\frac{c_3^2\,(c_1 + c_2)}{2} + c_1(1 - 2\,c_3)\right] x_n^{2(c_3 - 1)},$$
$$\kappa_3 = \frac{c_1 \ln^2 x_n}{x_n^2} - \frac{c_1 \ln x_n}{x_n^2} + \frac{c_2 - 2\,c_1}{4x_n^2},$$

$$\kappa_4 = c_3\,(c_1 - c_2)\,x_n^{c_3-1}\ln x_n + \left[2\,c_1 + \frac{c_3\,(c_2 - 7c_1)}{3}\right] x_n^{c_3-1},$$
$$\kappa_5 = (3\,c_1 - c_2)\,\frac{\ln x_n}{x_n} - \frac{4\,c_1 - c_2}{3\,x_n},$$
$$\kappa_6 = 2\,c_1\,(1 - c_3)\,x_n^{c_3-2}\ln x_n + (c_2\,c_3 - c_1)\,x_n^{c_3-2},$$
$$\chi_1 = \ln^2 x_n - \frac{8}{3}\ln x_n + \frac{16}{9}, \quad \chi_2 = x_n^{2(c_3-1)},$$
$$\chi_3 = \frac{\ln^2 x_n}{x_n^2} + \frac{\ln x_n}{x_n^2} + \frac{1}{4\,x_n^2}, \quad \chi_4 = \frac{8}{3}x_n^{c_3-1} - 2x_n^{c_3-1}\ln x_n,$$
$$\chi_5 = \frac{4}{3x_n} + \frac{5\ln x_n}{3x_n} - \frac{2\ln^2 x_n}{x_n}, \quad \chi_6 = 2\,x_n^{c_3-2}\ln x_n + x_n^{c_3-2}, \tag{5.6}$$

where γ_i $(i = 1, \ldots, 4)$ is given by Equation (2.22). With regard to Equations (4.7), (5.6), we get

$$\Phi_{1M} = \frac{c_{2M} - c_{1M}}{6}\left\{x_M^3\left[\left(\ln x_M - \frac{1}{3}\right)^2 + \frac{1}{9}\right] - x_{IN}^3\left[\left(\ln x_{IN} - \frac{1}{3}\right)^2 + \frac{1}{9}\right]\right\}$$
$$+ \frac{c_{1M} - c_{2M}}{9}\left[x_M^3\left(\ln x_M - \frac{1}{3}\right) - x_{IN}^3\left(\ln x_{IN} - \frac{1}{3}\right)\right] + \frac{17\,c_{1M} + c_{2M}}{54}\left(x_M^3 - x_{IN}^3\right),$$
$$\Phi_{2M} = \frac{1}{2\,c_{3M} + 1}\left[\frac{c_{3M}^2\,(c_{1M} + c_{2M})}{2} + c_{1M}\,(1 - 2\,c_{3M})\right]\left(x_M^{2\,c_{3M}+1} - x_{IN}^{2\,c_{3M}+1}\right),$$
$$\Phi_{3M} = c_{1M}\left[x_M\left(\ln^2 x_M - 2\ln x_M + 2\right) - x_{IN}\left(\ln^2 x_{IN} - 2\ln x_{IN} + 2\right)\right]$$
$$- c_{1M}\left[x_M\,(\ln x_M - 1) - x_{IN}\,(\ln x_{IN} - 1)\right] + \frac{c_{2M} - 2\,c_{1M}}{4}\,(x_M - x_{IN}),$$
$$\Phi_{4M} = \frac{c_{3M}\,(c_{1M} - c_{2M})}{c_{3M} + 2}$$
$$\times\left[x_M^{c_{3M}+2}\left(\ln x_M - \frac{1}{c_{3M}+2}\right) - x_{IN}^{c_{3M}+2}\left(\ln x_{IN} - \frac{1}{c_{3M}+2}\right)\right]$$
$$+ \frac{1}{c_{3M}+2}\left[2\,c_{1M} + \frac{c_{3M}\,(c_{2M} - 7c_{1M})}{3}\right]\left(x_M^{c_{3M}+2} - x_{IN}^{c_{3M}+2}\right),$$
$$\Phi_{5M} = \frac{3\,c_{1M} - c_{2M}}{2}\left[x_M^2\left(\ln x_M - \frac{1}{2}\right) - x_{IN}^2\left(\ln x_{IN} - \frac{1}{2}\right)\right]$$
$$- \frac{4\,c_{1M} - c_{2M}}{6}\left(x_M^2 - x_{IN}^2\right),$$
$$\Phi_{6M} = \frac{2\,c_{1M}\,(1 - c_{3M})}{c_{3M} + 1}\left[x_M^{c_{3M}+1}\left(\ln x_M - \frac{1}{c_{3M}+1}\right) - x_{IN}^{c_{3M}+1}\left(\ln x_{IN} - \frac{1}{c_{3M}+1}\right)\right]$$
$$+ \frac{c_{2M}\,c_{3M} - c_{1M}}{c_{3M}+1}\left(x_M^{c_{3M}+1} - x_{IN}^{c_{3M}+1}\right),$$

$$\Psi_{1M} = \frac{x_M^3}{3}\left[(\ln x_M - 3)\left(\ln x_M - \frac{1}{3}\right) + \frac{17}{9}\right]$$
$$- \frac{x_{IN}^3}{3}\left[(\ln x_{IN} - 3)\left(\ln x_{IN} - \frac{1}{3}\right) + \frac{17}{9}\right],$$
$$\Psi_{2M} = \frac{x_M^{2c_{3M}+1} - x_{IN}^{2c_{3M}+1}}{2c_{3M}+1},$$
$$\Psi_{3M} = x_M \ln x_M (\ln x_M - 1) - x_{IN} \ln x_{IN} (\ln x_{IN} - 1) + \frac{5(x_M - x_{IN})}{4},$$
$$\Psi_{4M} = \frac{2}{c_{3M}+2}\left\{x_M^{c_{3M}+2}\left[\frac{4c_{3M}+11}{3(c_{3M}+2)} - \ln x_M\right] - x_{IN}^{c_{3M}+2}\left[\frac{4c_{3M}+11}{3(c_{3M}+2)} - \ln x_{IN}\right]\right\},$$
$$\Psi_{5M} = \frac{2\left(x_M^2 - x_{IN}^2\right)}{3} + \frac{5}{6}\left[x_M^2\left(\ln x_M - \frac{1}{2}\right) - x_{IN}^2\left(\ln x_{IN} - \frac{1}{2}\right)\right]$$
$$- x_M^2\left(\ln^2 x_M - \ln x_M + \frac{1}{2}\right) + x_{IN}^2\left(\ln^2 x_{IN} - \ln x_{IN} + \frac{1}{2}\right),$$
$$\Psi_{6M} = \frac{2}{c_{3M}+1}\left[x_M^{c_{3M}+1}\left(\ln x_M - \frac{1}{c_{3M}+1}\right) - x_{IN}^{c_{3M}+1}\left(\ln x_{IN} - \frac{1}{c_{3M}+1}\right)\right]$$
$$+ \frac{1}{c_{3M}+1}\left(x_M^{c_{3M}+1} - x_{IN}^{c_{3M}+1}\right), \tag{5.7}$$

where x_{IN}, x_M are given by Equation (1.9), respectively. The integrals (4.14), which consider Equation (5.7), are determined by the formulae in Chapter 10 (see Equations (10.10)–(10.12)).

In case of the ellipsoidal inclusion, we get $(u_{nIN})_{x_n \to 0} \longrightarrow \pm\infty$, $(\varepsilon_{IN})_{x_n \longrightarrow 0} \longrightarrow \pm\infty$, $(\sigma_{IN})_{r \to 0} \longrightarrow \pm\infty$ due to $(\ln x_n)_{x_n \to 0} \longrightarrow \pm\infty$ and $(x_n^{c_3})_{x_n \to 0} \longrightarrow \pm\infty$ for $c_3 < 0$ (see Equations (2.18), (5.4)). Accordingly, the mathematical solutions (5.4) are suitable for the matrix.

5.2 Cell Matrix

The stress σ_{nB} is determined by two mathematical boundary conditions, i.e., by Equations (2.28), (2.29). With regard to three integration constants C_{1M}, C_{2M}, C_{3M} in Equation (5.4), the following conditions are considered to determine σ_{nB}, i.e., $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} = 0$; $C_{1M} \neq 0$, $C_{3M} \neq 0$, $C_{2M} = 0$; $C_{2M} \neq 0$, $C_{3M} \neq 0$, $C_{1M} = 0$. Consequently, the mathematical boundary conditions (2.31)–(2.33) are applied in case of $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} \neq 0$.

Conditions $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} = 0$. With regard to Equations (2.28), (2.29), (2.35), (5.4), (5.5), we get

$$
\begin{aligned}
\varepsilon_{nM} &= -\frac{p_n}{\zeta_M}\left[\frac{1}{3} - \ln x_n - c_{3M}\left(\frac{4}{3} - \ln x_M\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right],\\
\varepsilon_{\varphi M} = \varepsilon_{\theta M} &= -\frac{p_n}{\zeta_M}\left[\frac{4}{3} - \ln x_n - \left(\frac{4}{3} - \ln x_M\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right],\\
\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} &= \left(\ln x_n - \frac{4}{3}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M}\right)\\
&\quad + x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta_M\, x_M^{c_{3M}-1}}\left(\frac{4}{3} - \ln x_M\right)\right],\\
\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} &= \Theta\left\{\left(\ln x_n - \frac{4}{3}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M}\right)\right.\\
&\quad \left. + x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta_M\, x_M^{c_{3M}-1}}\left(\frac{4}{3} - \ln x_M\right)\right]\right\},\\
\sigma_{nM} &= \frac{p_n}{\zeta_M}\left\{\frac{7c_{2M} - c_{1M}}{3} + (c_{1M} - c_{2M})\ln x_n\right.\\
&\quad \left. + \left[(c_{1M} + c_{2M})\,c_{3M} - 2c_{2M}\right]\left(\frac{4}{3} - \ln x_M\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right\},\\
\sigma_{\varphi M} = \sigma_{\theta M} &= \frac{p_n}{\zeta_M}\left[\frac{c_{2M} - 4c_{1M}}{3} + (c_{1M} - c_{2M})\ln x_n\right.\\
&\quad \left. + (c_{1M} - c_{2M}\,c_{3M})\left(\frac{4}{3} - \ln x_M\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right],\\
\sigma_{nB} &= -\frac{\rho_{1B}\,p_n}{\zeta_M} - \frac{1}{s_{44M}}\left[\rho_{1B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M}\right) + \rho_{1B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M}\right)\right]\\
&\quad + \frac{\rho_{2B}\,p_n}{\zeta_M x_M^{c_{3M}-1}}\left(\frac{4}{3} - \ln x_M\right) + \frac{\rho_{2B}^{(\varphi)}}{s_{44M}}\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta_M x_M^{c_{3M}-1}}\left(\frac{4}{3} - \ln x_M\right)\right]\\
&\quad + \frac{\rho_{2B}^{(\nu)}}{s_{44M}}\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta_M x_M^{c_{3M}-1}}\left(\frac{4}{3} - \ln x_M\right)\right],
\end{aligned}
\tag{5.8}
$$

where Θ, x_M, s_{44M}, $c_{iM}(i=1,2,3)$ are given by Equations (1.8), (1.9), (2.13), (2.18), respectively. The coefficients ζ_M, ζ_{iM}, ρ_{iB}, $\rho_{iB}^{(\tau)}$ $(i=1,2;\ \tau=\varphi,\nu)$ have the forms

$$
\zeta_M = \zeta_{2M} - \zeta_{1M}\left(\frac{4}{3} - \ln x_M\right),
$$

$$\zeta_{1M} = \zeta_{1MB}\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1}, \quad \zeta_{2M} = \frac{c_{1M}-7\,c_{2M}}{3} - (c_{1M}-c_{2M})\ln x_{IN},$$
$$\zeta_{1MB} = (c_{1M}+c_{2M})\,c_{3M} - 2\,c_{2M}, \quad \zeta_{2MB} = \frac{c_{1M}-7c_{2M}}{3} - (c_{1M}-c_{2M})\ln x_M,$$
$$\rho_{1B} = \frac{1}{3}\Big\{\gamma_{1M}(c_{1M}-7\,c_{2M}) + (\gamma_{2M}+\gamma_{3M})(4\,c_{1M}-c_{2M})\Big\}$$
$$+ (c_{2M}-c_{1M})(\gamma_{1M}+\gamma_{2M}+\gamma_{3M})\ln x_M,$$
$$\rho_{2B} = \Big\{\gamma_{1M}[(c_{1M}+c_{2M})\,c_{3M}-2\,c_{2M}] + (\gamma_{2M}+\gamma_{3M})(c_{1M}-c_{2M}\,c_{3M})\Big\}x_M^{c_{3M}-1},$$
$$\rho_{1B}^{(\varphi)} = (\gamma_{1M}+\gamma_{2M})\left(\frac{4}{3}-\ln x_M\right), \quad \rho_{1B}^{(\nu)} = \Theta(\gamma_{1M}+\gamma_{3M})\left(\frac{4}{3}-\ln x_M\right),$$
$$\rho_{2B}^{(\varphi)} = (\gamma_{1M}+\gamma_{2M})\,x_M^{c_{3M}-1}, \quad \rho_{2B}^{(\nu)} = \Theta(\gamma_{1M}+\gamma_{3M})\,x_M^{c_{3M}-1}, \tag{5.9}$$

where γ_{iM}, ϑ_j ($i=1,\ldots,4;\ j=1,2,3$) are given by Equations (2.22), (2.32), respectively. The normal stress p_n is given by Equations (2.30) or (2.34) for $N=1$ or $N+1=1,2,\ldots$, respectively. With regard to Equation (5.8) and $\left[\varepsilon_{\varphi M}^{(1)}\right]_{x_n=x_M} = -p_n^{(1)}\rho_M$ [1]–[22], the coefficient ρ_M in Equation (2.30) for $N=1$ is derived as

$$\rho_M = \frac{1}{\zeta_M}\left[\frac{4}{3}-\ln x_{IN} - \left(\frac{4}{3}-\ln x_M\right)\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1}\right], \quad Q = M, MB. \tag{5.10}$$

Conditions $C_{1M}\neq 0$, $C_{3M}\neq 0$, $C_{2M}=0$. With regard to Equations (2.28), (2.29), (2.35), (5.4), (5.5), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta_M}\left[\left(\frac{1}{2}+\ln x_M\right)\left(\frac{1}{3}-\ln x_n\right) - \frac{x_M}{x_n}\left(\frac{4}{3}-\ln x_M\right)\right],$$
$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta_M}\left[\left(\frac{1}{2}+\ln x_M\right)\left(\frac{4}{3}-\ln x_n\right) - \frac{x_M}{x_n}\left(\frac{4}{3}-\ln x_M\right)\left(\frac{1}{2}+\ln x_n\right)\right],$$
$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = \left(\ln x_n - \frac{4}{3}\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta_M}\left(\frac{1}{2}+\ln x_M\right)\right]$$
$$+ \frac{1}{x_n}\left(\frac{1}{2}+\ln x_n\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n x_M}{\zeta_M}\left(\ln x_M - \frac{4}{3}\right)\right],$$
$$\varepsilon_{n\nu M} = s_{44M}\,\sigma_{n\varphi M} = \Theta\Bigg\{\left(\ln x_n - \frac{4}{3}\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta_M}\left(\frac{1}{2}+\ln x_M\right)\right]$$
$$+ \frac{1}{x_n}\left(\frac{1}{2}+\ln x_n\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n x_M}{\zeta_M}\left(\ln x_M - \frac{4}{3}\right)\right]\Bigg\},$$

$$
\begin{aligned}
\sigma_{nM} = \frac{p_n}{\zeta_M} \Bigg\{ & \left(\frac{1}{2} + \ln x_M\right) \left[\frac{c_{1M} - 7c_{2M}}{3} - (c_{1M} - c_{2M}) \ln x_n\right] \\
& + \frac{x_M}{x_n} \left(\frac{4}{3} - \ln x_M\right) (c_{1M} - 2c_{2M} \ln x_n) \Bigg\}, \\
\sigma_{\varphi M} = \sigma_{\theta M} = \frac{p_n}{\zeta_M} \Bigg\{ & \left(\frac{1}{2} + \ln x_M\right) \left[\frac{4c_{1M} - c_{2M}}{3} - (c_{1M} - c_{2M}) \ln x_n\right] \\
& + \frac{x_M}{x_n} \left(\frac{4}{3} - \ln x_M\right) \left(\frac{c_{1M} - c_{2M}}{2} + c_{3M} \ln x_n\right) \Bigg\}, \\
\sigma_{nB} = & -\frac{\rho_{1B} p_n}{\zeta_M} \left(\frac{1}{2} + \ln x_M\right) - \frac{\rho_{3B} p_n x_M}{\zeta_M} \left(\frac{4}{3} - \ln x_M\right) \\
& - \frac{1}{s_{44M}} \left\{ \rho_{1B}^{(\varphi)} \frac{\partial}{\partial \varphi} \left[\frac{p_n}{\zeta_M} \left(\frac{1}{2} + \ln x_M\right)\right] + \rho_{3B}^{(\nu)} \frac{\partial}{\partial \nu} \left[\frac{p_n}{\zeta_M} \left(\frac{1}{2} + \ln x_M\right)\right] \right\} \\
& - \frac{1}{s_{44M}} \left\{ \rho_{3B}^{(\varphi)} \frac{\partial}{\partial \varphi} \left[\frac{p_n x_M}{\zeta_M} \left(\frac{4}{3} - \ln x_M\right)\right] + \rho_{3B}^{(\nu)} \frac{\partial}{\partial \nu} \left[\frac{p_n x_M}{\zeta_M} \left(\frac{4}{3} - \ln x_M\right)\right] \right\},
\end{aligned}
\tag{5.11}
$$

where Θ, x_M, s_{44M}, c_{iM}; ρ_{1B}, $\rho_{1B}^{(\tau)}$ $(i = 1,2,3\ \tau = \varphi, \nu)$ are given by Equations (1.8), (1.9), (2.13), (2.18); (5.9), respectively. The coefficients ζ_M, ζ_{iM}, ρ_{3B}, $\rho_{3B}^{(\tau)}$ $(i = 1,2;$ $\tau = \varphi, \nu)$ have the forms

$$
\begin{aligned}
& \zeta_M = \frac{\zeta_{2M}}{x_M} \left(\frac{1}{2} + \ln x_M\right) - \zeta_{1M} \left(\frac{4}{3} - \ln x_M\right), \\
& \zeta_{1M} = \frac{x_M}{x_{IN}} (c_{1M} - 2c_{2M} \ln x_{IN}), \quad \zeta_{2M} = x_M \left[\frac{c_{1M} - 7c_{2M}}{3} - (c_{1M} - c_{2M}) \ln x_{IN}\right], \\
& \zeta_{1MB} = c_{1M} - 2c_{2M} \ln x_M, \quad \zeta_{2MB} = x_M \left[\frac{c_{1M} - 7c_{2M}}{3} - (c_{1M} - c_{2M}) \ln x_M\right], \\
& \rho_{3B} = \frac{1}{x_M} \Bigg\{ c_{1M}\gamma_{1M} + \frac{(c_{1M} - 2c_{2M})(\gamma_{2M} + \gamma_{3M})}{2} \\
& \qquad + [c_{1M}(\gamma_{2M} + \gamma_{3M}) - 2c_{2M}\gamma_{1M}] \ln x_M \Bigg\}, \\
& \rho_{3B}^{(\varphi)} = \frac{(\gamma_{1M} + \gamma_{2M})}{x_M} \left(\frac{1}{2} + \ln x_M\right), \quad \rho_{3B}^{(\nu)} = \frac{\Theta(\gamma_{1M} + \gamma_{3M})}{x_M} \left(\frac{1}{2} + \ln x_M\right),
\end{aligned}
\tag{5.12}
$$

where γ_{iM}, ϑ_j $(i = 1, \ldots, 4;\ j = 1,2,3)$ are given by Equations (2.22), (2.32), respectively. The normal stress p_n is given by Equations (2.30) or (2.34) for $N = 1$ or $N + 1 = 1, 2, \ldots$, respectively. With regard to Equation (5.11) and $\left[\varepsilon_{\varphi M}^{(1)}\right]_{x_n = x_M} =$

$-p_n^{(1)}\,\rho_M$ [1]–[22], the coefficient ρ_M in Equation (2.30) for $N=1$ is derived as

$$\rho_M = \frac{1}{\zeta_M}\left[\left(\frac{1}{2}+\ln x_M\right)\left(\frac{4}{3}-\ln x_{IN}\right)-\frac{x_M}{x_{IN}}\left(\frac{4}{3}-\ln x_M\right)\left(\frac{1}{2}+\ln x_{IN}\right)\right]. \quad (5.13)$$

Conditions $C_{2M}\neq 0$**,** $C_{3M}\neq 0$**,** $C_{1M}=0$**.** With regard to Equations (2.28), (2.29), (2.35), (5.4), (5.5), we get

$$\begin{aligned}
\varepsilon_{nM} &= -\frac{p_n}{\zeta_M}\left[c_{3M}\left(\frac{1}{2}+\ln x_M\right)x_n^{c_{3M}-1}-\frac{x_M^{c_{3M}}}{x_n}\right],\\
\varepsilon_{\varphi M} = \varepsilon_{\theta M} &= -\frac{p_n}{\zeta_M}\left[\left(\frac{1}{2}+\ln x_M\right)x_n^{c_{3M}-1}-\frac{x_M^{c_{3M}}}{x_n}\left(\frac{1}{2}+\ln x_n\right)\right],\\
\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} &= \frac{1}{x_n}\left(\frac{1}{2}+\ln x_n\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^{c_{3M}}}{\zeta_M}\right)\\
&\quad - x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta_M}\left(\frac{1}{2}+\ln x_M\right)\right],\\
\varepsilon_{n\nu M} = s_{44M}\,\sigma_{n\nu M} &= \Theta\left\{\frac{1}{x_n}\left(\frac{1}{2}+\ln x_n\right)\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^{c_{3M}}}{\zeta_M}\right)\right.\\
&\quad \left.- x_n^{c_{3M}-1}\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta_M}\left(\frac{1}{2}+\ln x_M\right)\right]\right\},\\
\sigma_{nM} &= -\frac{p_n x_n^{c_{3M}-1}}{\zeta_M}\\
&\quad \times\left\{\left[c_{3M}\left(c_{1M}+c_{2M}\right)-2c_{2M}\right]\left(\frac{1}{2}+\ln x_M\right)-\frac{x_M}{x_n}\left(c_{1M}-2c_{2M}\ln x_n\right)\right\},\\
\sigma_{\varphi M} = \sigma_{\theta M} &=\\
&-\frac{p_n x_n^{c_{3M}-1}}{\zeta_M}\left[\left(c_{1M}-c_{2M}c_{3M}\right)\left(\frac{1}{2}+\ln x_M\right)-\frac{x_M}{x_n}\left(\frac{c_{1M}-2c_{2M}}{2}+c_{1M}\ln x_n\right)\right],\\
\sigma_{nB} &= -\frac{\rho_{1B}\,p_n}{\zeta_M}\left(\frac{1}{2}+\ln x_M\right)+\frac{\rho_{3B}\,p_n x_M^{c_{3M}}}{\zeta_M}\\
&\quad -\frac{1}{s_{44M}}\left\{\rho_{1B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta_M}\left(\frac{1}{2}+\ln x_M\right)\right]+\rho_{3B}^{(\nu)}\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta_M}\left(\frac{1}{2}+\ln x_M\right)\right]\right\}\\
&\quad +\frac{1}{s_{44M}}\left[\rho_{3B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^{c_{3M}}}{\zeta_M}\right)+\rho_{3B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^{c_{3M}}}{\zeta_M}\right)\right],
\end{aligned} \quad (5.14)$$

where Θ, x_M, s_{44M}, c_{iM} $(i=1,2,3)$; ρ_{jB}, $\rho_{jB}^{(\tau)}$ $(j=2,3;\ \tau=\varphi,\nu)$ are given by Equations (1.8), (1.9), (2.13), (2.18); (5.9), (5.12), respectively. The coefficients ζ_M, ζ_{iM}

($i=1,2$) have the forms

$$\zeta_M = \frac{\zeta_{2M}}{x_M}\left(\frac{1}{2}+\ln x_M\right) - \zeta_{1M}\,x_M^{c_{3M}-1},$$
$$\zeta_{1M} = \frac{x_M}{x_{IN}}\left(c_{1M}-2c_{2M}\ln x_{IN}\right), \quad \zeta_{2M} = x_M\left[c_{3M}\left(c_{1M}+c_{2M}\right)-2c_{2M}\right]x_{IN}^{c_{3M}-1},$$
$$\zeta_{1MB} = c_{1M}-2c_{2M}\ln x_M, \quad \zeta_{2MB} = \left[c_{3M}\left(c_{1M}+c_{2M}\right)-2c_{2M}\right]x_M^{c_{3M}}, \tag{5.15}$$

where γ_{iM} ($i=1,\ldots,4$) is given by Equation (2.22). The normal stress p_n is given by Equations (2.30) or (2.34) for $N=1$ or $N+1=1,2,\ldots$, respectively. With regard to Equation (5.14) and $\left[\varepsilon_{\varphi M}^{(1)}\right]_{x_n=x_M} = -p_n^{(1)}\rho_M$ [1]–[22], the coefficient ρ_M in Equation (2.30) for $N=1$ is derived as

$$\rho_M = \frac{1}{\zeta_M}\left[\left(\frac{1}{2}+\ln x_M\right)x_{IN}^{c_{3M}-1} - \frac{x_M^{c_{3M}}}{x_{IN}}\left(\frac{1}{2}+\ln x_{IN}\right)\right]. \tag{5.16}$$

Conditions $C_{1M}\neq 0$, $C_{2M}\neq 0$, $C_{3M}\neq 0$. Finally, with regard to Equations (2.21), (2.27), (2.31)–(2.33), (5.4), (5.5), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta_M}\left[\zeta_{1M}\left(\frac{1}{3}-\ln x_n\right)+\zeta_{2M}\,c_{3M}\,x_n^{c_{3M}-1}+\frac{\zeta_{3M}}{x_n}\right],$$
$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta_M}\left[\zeta_{1M}\left(\frac{4}{3}-\ln x_n\right)+\zeta_{2M}\,x_n^{c_{3M}-1}+\frac{\zeta_{3M}}{x_n}\left(\frac{1}{2}+\ln x_n\right)\right],$$
$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = -\left[\left(\frac{4}{3}-\ln x_n\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_{1M}}{\zeta_M}\right)+x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_{2M}}{\zeta_M}\right)\right.$$
$$\left.+\frac{1}{x_n}\left(\frac{1}{2}+\ln x_n\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_{3M}}{\zeta_M}\right)\right],$$
$$\varepsilon_{n\nu M} = s_{44M}\,\sigma_{n\varphi M} = -\Theta\left[\left(\frac{4}{3}-\ln x_n\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_{1M}}{\zeta_M}\right)+x_n^{c_{3M}-1}\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_{2M}}{\zeta_M}\right)\right.$$
$$\left.+\frac{1}{x_n}\left(\frac{1}{2}+\ln x_n\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_{3M}}{\zeta_M}\right)\right],$$
$$\sigma_{nM} = -\frac{p_n}{\zeta_M}\left\{\zeta_{1M}\left[\frac{c_{1M}-7c_{2M}}{3}-\left(c_{1M}-c_{2M}\right)\ln x_n\right]\right.$$
$$\left.+\zeta_{2M}\left[\left(c_{1M}+c_{2M}\right)c_{3M}-2c_{2M}\right]x_n^{c_{3M}-1}+\frac{\zeta_{3M}\left(c_{1M}-2c_{2M}\ln x_n\right)}{x_n}\right\},$$
$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta_M}\left\{\zeta_{1M}\left[\frac{4c_{1M}-c_{2M}}{3}-\left(c_{1M}-c_{2M}\right)\ln x_n\right]\right.$$

$$+\,\zeta_{2M}\,(c_{1M}-c_{2M}\,c_{3M})\,x_n^{c_{3M}-1}+\zeta_{3M}\left(\frac{c_{1M}-2\,c_{2M}}{2}+c_{1M}\ln x_n\right)\Bigg\},$$

$$\sigma_{1M}=\eta_{1M}+\eta_{2M}\ln x_n+\eta_{3M}\,x_n^{c_{3M}-1}+\frac{\eta_{4M}+\eta_{5M}\ln x_n}{x_n},$$

$$w_M=\left(\frac{p_n}{\zeta_M}\right)^2\Big(\kappa_{1M}\,\zeta_{1M}^2+\kappa_{2M}\,\zeta_{2M}^2+\kappa_{3M}\,\zeta_{3M}^2$$
$$+\,\kappa_{4M}\,\zeta_{1M}\,\zeta_{2M}+\kappa_{5M}\,\zeta_{1M}\,\zeta_{3M}+\kappa_{6M}\,\zeta_{2M}\,\zeta_{3M}\Big)$$
$$+\,\frac{\chi_{1M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{1M}}{\zeta_M}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{1M}}{\zeta_M}\right)\right]^2\right\}$$
$$+\,\frac{\chi_{2M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{2M}}{\zeta_M}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{2M}}{\zeta_M}\right)\right]^2\right\}$$
$$+\,\frac{\chi_{3M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{3M}}{\zeta_M}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{3M}}{\zeta_M}\right)\right]^2\right\}$$
$$+\,\frac{\chi_{4M}}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{1M}}{\zeta_M}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{2M}}{\zeta_M}\right)+\Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{1M}}{\zeta_M}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{2M}}{\zeta_M}\right)\right]$$
$$+\,\frac{\chi_{5M}}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{1M}}{\zeta_M}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{3M}}{\zeta_M}\right)+\Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{1M}}{\zeta_M}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{3M}}{\zeta_M}\right)\right]$$
$$+\,\frac{\chi_{6M}}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{2M}}{\zeta_M}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{3M}}{\zeta_M}\right)+\Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{2M}}{\zeta_M}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{3M}}{\zeta_M}\right)\right],$$

$$W_M=4\int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{p_n}{\zeta_M}\right)^2\Big(\Phi_{1M}\,\zeta_{1M}^2+\Phi_{2M}\,\zeta_{2M}^2+\Phi_{3M}\,\zeta_{3M}^2+\Phi_{4M}\,\zeta_{1M}\,\zeta_{2M}$$
$$+\,\Phi_{5M}\,\zeta_{1MM}\,\zeta_{3M}+\Phi_{6M}\,\zeta_{2M}\,\zeta_{3M}\Big)\,d\varphi\,d\nu$$
$$+\,\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{1M}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{1M}}{\zeta_M}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{1M}}{\zeta_M}\right)\right]^2\right\}d\varphi\,d\nu$$
$$+\,\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{2M}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{2M}}{\zeta_M}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{2M}}{\zeta_M}\right)\right]^2\right\}d\varphi\,d\nu$$
$$+\,\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{3M}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{3M}}{\zeta_M}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{3M}}{\zeta_M}\right)\right]^2\right\}d\varphi\,d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{4M}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_{1M}}{\zeta_M}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_{2M}}{\zeta_M}\right)\right.$$
$$\left.+\Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_{1M}}{\zeta_M}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_{2M}}{\zeta_M}\right)\right]d\varphi\,d\nu$$
$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{5M}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_{1M}}{\zeta_M}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_{3M}}{\zeta_M}\right)\right.$$
$$\left.+\Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_{1M}}{\zeta_M}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_{3M}}{\zeta_M}\right)\right]d\varphi\,d\nu$$
$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{6M}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_{2M}}{\zeta_M}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_{3M}}{\zeta_M}\right)\right.$$
$$\left.+\Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_{2M}}{\zeta_M}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_{3M}}{\zeta_M}\right)\right]d\varphi\,d\nu,$$
$$\sigma_{nB}=-\sum_{i=1}^{3}\frac{\rho_{iB}\,p_n\zeta_{iM}}{\zeta_M}+\frac{1}{s_{44M}}\left[\rho_{iB}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_{iM}}{\zeta_M}\right)+\rho_{iB}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_{iM}}{\zeta_M}\right)\right],\qquad(5.17)$$

where Θ, x_M, s_{44M}, c_{iM}; κ_{jM}, χ_{jM}; Φ_{jM}, Ψ_{jM}; ρ_{iB}, $\rho_{iB}^{(\tau)}$ ($i=1,2,3$; $j=1,\dots,6$; $\tau=\varphi,\nu$); are given by Equations (1.8), ((1.9), (2.13), (2.18); (5.6); (5.7); (5.10), (5.13); respectively. The coefficients ζ_{iM}, ζ_M, η_{jM} ($i=1,2,3$; $j=1,\dots,4$; see Equation (5.6)) have the forms

$$\zeta_{iM}=-p_n\zeta_{i1M}+\sigma_{nB}\zeta_{i2M},\quad i=1,2,3,$$
$$\zeta_{11M}=(c_{1M}+c_{2M})\left[c_{3M}\left(\frac{1}{2}+\ln x_M\right)-1\right]x_M^{c_{3M}-1},$$
$$\zeta_{12M}=[(c_{1M}+c_{2M})\,c_{3M}-2c_{2M}]\left(\frac{1}{2}+\ln x_M\right)x_{IN}^{c_{3M}-1}-\frac{(c_{1M}-c_{2M}\ln x_{IN})\,x_M^{c_{3M}}}{x_{IN}},$$
$$\zeta_{21M}=(c_{1M}+c_{2M})\left(\frac{7-5\ln x_M}{6}+\ln^2 x_M\right),$$
$$\zeta_{22M}=(c_{1M}-2\,c_{2M}\ln x_{IN})\left(\frac{4}{3}-\ln x_M\right)\frac{x_M}{x_{IN}}$$
$$-\left[\frac{c_{1M}-7\,c_{2M}}{3}-(c_{1M}-c_{2M})\ln x_{IN}\right]\left(\frac{1}{2}+\ln x_M\right),$$
$$\zeta_{31M}=(c_{1M}+c_{2M})\left[\frac{1-4\,c_{3M}}{3}+(c_{3M}-1)\ln x_M\right]x_M^{c_{3M}},$$
$$\zeta_{32M}=\left[\frac{c_{1M}-7\,c_{2M}}{3}-(c_{1M}-c_{2M})\ln x_{IN}\right]x_M^{c_{3M}},$$

$$
\begin{aligned}
&\quad - \left[(c_{1M}+c_{2M})\,c_{3M}-2\,c_{2M}\right]\left(\frac{4}{3}-\ln x_M\right)x_M\,x_{IN}^{c_{3M}-1},\\
\zeta_M &= (c_{1M}+c_{2M})\\
&\quad\times\left\{\left[\frac{c_{1M}-7\,c_{2M}}{3}-(c_{1M}-c_{2M})\ln x_M\right]\left[c_{3M}\left(\frac{1}{2}+\ln x_M\right)-1\right]x_M^{c_{3M}-1}\right.\\
&\quad +\left[(c_{1M}+c_{2M})\,c_{3M}-2\,c_{2M}\right]\left[7\,(c_{1M}+c_{2M})-5\ln x_M+6\ln^2 x_M\right]\frac{x_{IN}^{c_{3M}-1}}{6}\\
&\quad \left.+(c_{1M}-2\,c_{2M}\ln x_{IN})\left[\frac{1-4\,c_{3M}}{3}+(c_{3M}-1)\ln x_M\right]\frac{x_M^{c_{3M}}}{x_{IN}}\right\},\\
\eta_{1M} &= -\frac{p_n\,\zeta_{1M}\,(\gamma_{1M}+4\,\gamma_{2M})}{3\,\zeta_M}\\
&\quad -\frac{4}{3\,s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{1M}}{\zeta_M}\right)+\gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{1M}}{\zeta_M}\right)\right],\\
\eta_{2M} &= \frac{p_n\,\zeta_{1M}\,(\gamma_{1M}+\gamma_{2M})}{\zeta_M}\\
&\quad +\frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{1M}}{\zeta_M}\right)+\gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{1M}}{\zeta_M}\right)\right],\\
\eta_{3M} &= -\frac{p_n\,\zeta_{2M}\,(\gamma_{1M}\,c_{3M}+\gamma_{2M})}{\zeta_M}\\
&\quad -\frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{2M}}{\zeta_M}\right)+\gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{2M}}{\zeta_M}\right)\right],\\
\eta_{4M} &= -\frac{p_n\,\zeta_{3M}\,(2\,\gamma_{1M}+\gamma_{2M})}{2\,\zeta_M}\\
&\quad -\frac{1}{2\,s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{3M}}{\zeta_M}\right)+\gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{3M}}{\zeta_M}\right)\right],\\
\eta_{5M} &= -\frac{p_n\,\zeta_{3M}\,\gamma_{2M}}{\zeta_M}\,\frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{3M}}{\zeta_M}\right)+\gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{3M}}{\zeta_M}\right)\right],
\end{aligned}
\tag{5.18}
$$

where γ_{iM} $(i=1,\ldots,4)$ is given by Equation (2.22). The normal stress p_n is given by Equations (2.30) or (2.34) for $N=1$ or $N+1=1,2,\ldots$, respectively. With regard to Equation (5.17) and $\left[\varepsilon_{\varphi M}\right]_{x_n=x_M}=-p_n\,\phi_M+\sigma_{nB}\,\phi_B$, the coefficients ϕ_M, ϕ_B in Equation (2.34) for $N+1=1,2,\ldots$ are derived as

$$
\begin{aligned}
\phi_M &= \frac{1}{\zeta_M}\left[\zeta_{11M}\left(\frac{4}{3}-\ln x_{IN}\right)+\zeta_{21M}\,x_{IN}^{c_{3M}-1}+\frac{\zeta_{31M}}{x_{IN}}\left(\frac{1}{2}+\ln x_{IN}\right)\right],\\
\phi_B &= \frac{1}{\zeta_M}\left[\zeta_{12M}\left(\frac{4}{3}-\ln x_{IN}\right)+\zeta_{22M}\,x_{IN}^{c_{3M}-1}+\frac{\zeta_{32M}}{x_{IN}}\left(\frac{1}{2}+\ln x_{IN}\right)\right].
\end{aligned}
\tag{5.19}
$$

Chapter 6

Mathematical Model 4

6.1 Mathematical Procedure

The differential equation (2.23) is transformed to the form

$$U_n = -s_{44}\left(c_1 + c_2\right)\left(x_n^2 \frac{\partial^2 u_n}{\partial x_n^2} + 2x_n \frac{\partial u_n}{\partial x_n} - 2u_n\right), \tag{6.1}$$

where s_{44}, c_i $(i = 1,2)$ and $U_n = U_n\left(x_n, \varphi, \nu\right)$ are given by Equations (2.13), (2.18) and (2.25), respectively. Let $x_n\left[\partial \mathrm{Eq.}(6.1)/\partial x_n\right]$ be performed, and then we get

$$x_n \frac{\partial U_n}{\partial x_n} = -s_{44}\left(c_1 + c_2\right)\left(x_n^3 \frac{\partial^3 u_n}{\partial x_n^3} + 4x_n^2 \frac{\partial^2 u_n}{\partial x_n^2}\right). \tag{6.2}$$

Let Equations (6.1), (6.2 be substituted to Equation (2.24, and then we get

$$x_n^3 \frac{\partial^3 u_n}{\partial x_n^3} + \left(4 - c_3\right) x_n^2 \frac{\partial^2 u_n}{\partial x_n^2} - 2c_3 x_n \frac{\partial u_n}{\partial x_n} + 2c_3 u_n = 0. \tag{6.3}$$

Let u_n be assumed in the form $u_n = x_n^{\lambda}$, then we get [1]–[22]

$$u_n = C_1 x_n + C_2 x_n^{c_3} + \frac{C_3}{x_n^2}, \tag{6.4}$$

where $c_3 < 0$ is given by Equation (2.18), and C_1, C_2, C_3 are integration constants, which are determined by the mathematical boundary conditions in Section 2.3. With regard to Equations (2.1)–(2.4), (2.14)–(2.17), (2.26), (6.6), we get

$$\varepsilon_n = C_1 + C_2 c_3 x_n^{c_3 - 1} - \frac{2C_3}{x_n^3},$$
$$\varepsilon_\varphi = \varepsilon_\theta = C_1 + C_2 x_n^{c_3 - 1} + \frac{C_3}{x_n^3},$$
$$\varepsilon_{n\varphi} = s_{44}\,\sigma_{n\varphi} = \frac{\partial C_1}{\partial \varphi} + x_n^{c_3 - 1}\frac{\partial C_2}{\partial \varphi} + \frac{1}{x_n^3}\frac{\partial C_3}{\partial \varphi},$$
$$\varepsilon_{n\theta} = s_{44}\,\sigma_{n\theta} = \Theta\left[\frac{\partial C_1}{\partial \nu} + x_n^{c_3 - 1}\frac{\partial C_2}{\partial \nu} + \frac{1}{x_n^3}\frac{\partial C_3}{\partial \nu}\right],$$

$$\sigma_n = C_1(c_1 - c_2) + C_2[(c_1 + c_2)c_3 - 2c_2]x_n^{c_3-1} - \frac{2C_3(c_1 + 2c_2)}{x_n^3},$$
$$\sigma_\varphi = \sigma_\theta = C_1(c_1 - c_2) + C_2(c_1 - c_2 c_3)x_n^{c_3-1} + \frac{C_3(c_1 + 2c_2)}{x_n^3},$$
$$\sigma_1 = \eta_1 + \eta_2 x_n^{c_3-1} + \frac{\eta_3}{x_n^3},$$
$$w = \kappa_1 + \kappa_2 x_n^{2(c_3-1)} + \frac{\kappa_3}{x_n^6} + \kappa_4 x_n^{c_3-1} + \frac{\kappa_5}{x_n^3} + \kappa_6 x_n^{c_3-4}, \tag{6.5}$$

where Θ is given by Equation (1.8). The coefficients η, κ_j ($i=1,2,3;\ j=1,\ldots,6$) are derived as

$$\eta_1 = C_1(\gamma_1 + \gamma_2) + \frac{1}{s_{44}}\left(\gamma_3 \frac{\partial C_1}{\partial \varphi} + \gamma_4 \frac{\partial C_1}{\partial \nu}\right),$$
$$\eta_2 = C_2(\gamma_1 c_3 + \gamma_2) + \frac{1}{s_{44}}\left(\gamma_3 \frac{\partial C_2}{\partial \varphi} + \gamma_4 \frac{\partial C_2}{\partial \nu}\right),$$
$$\eta_3 = C_3(\gamma_2 - 2\gamma_1) + \frac{1}{s_{44}}\left(\gamma_3 \frac{\partial C_3}{\partial \varphi} + \gamma_4 \frac{\partial C_3}{\partial \nu}\right),$$
$$\kappa_1 = \frac{3(c_1 - c_2)C_1^2}{2} + \frac{1}{s_{44}}\left[\left(\frac{\partial C_1}{\partial \varphi}\right)^2 + \Theta^2\left(\frac{\partial C_1}{\partial \nu}\right)^2\right],$$
$$\kappa_2 = \left[\frac{(c_1 + c_2)c_3^2}{2} + c_1 - 2c_2 c_3\right]C_2^2 + \frac{1}{s_{44}}\left[\left(\frac{\partial C_2}{\partial \varphi}\right)^2 + \Theta^2\left(\frac{\partial C_2}{\partial \nu}\right)^2\right],$$
$$\kappa_3 = 3(c_1 + 2c_2)C_3^2 + \frac{1}{s_{44}}\left[\left(\frac{\partial C_3}{\partial \varphi}\right)^2 + \Theta^2\left(\frac{\partial C_3}{\partial \nu}\right)^2\right],$$
$$\kappa_4 = (c_1 - c_2)(2 + c_3)C_1 C_2 + \frac{2}{s_{44}}\left(\frac{\partial C_1}{\partial \varphi}\frac{\partial C_2}{\partial \varphi} + \Theta^2 \frac{\partial C_1}{\partial \nu}\frac{\partial C_2}{\partial \nu}\right),$$
$$\kappa_5 = \frac{2}{s_{44}}\left(\frac{\partial C_1}{\partial \varphi}\frac{\partial C_3}{\partial \varphi} + \Theta^2 \frac{\partial C_1}{\partial \nu}\frac{\partial C_3}{\partial \nu}\right),$$
$$\kappa_6 = [2c_2(1 - c_3) - c_1]C_2 C_3 + \frac{2}{s_{44}}\left(\frac{\partial C_2}{\partial \varphi}\frac{\partial C_3}{\partial \varphi} + \Theta^2 \frac{\partial C_2}{\partial \nu}\frac{\partial C_3}{\partial \nu}\right), \tag{6.6}$$

where γ_i ($i=1,\ldots,4$) is given by Equation (2.22). In case of the ellipsoidal inclusion, we get $(u_{nIN})_{x_n \to 0} \longrightarrow \pm\infty$, $(\varepsilon_{IN})_{x_n \longrightarrow 0} \longrightarrow \pm\infty$, $(\sigma_{IN})_{r \to 0} \longrightarrow \pm\infty$ due to $(\ln x_n)_{x_n \to 0} \longrightarrow \pm\infty$ and $(x_n^{c_3})_{x_n \to 0} \longrightarrow \pm\infty$ for $c_3 < 0$ (see Equations (2.18), (6.4)). Accordingly, the mathematical solution (6.4) is suitable for the matrix.

6.2 Cell Matrix

The stress σ_{nB} is determined by two mathematical boundary conditions, i.e., by Equations (2.28), (2.29). With regard to three integration constants C_{1M}, C_{2M}, C_{3M} in Equation (6.4), the following conditions are considered to determine σ_{nB}, i.e., $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} = 0$; $C_{1M} \neq 0$, $C_{3M} \neq 0$, $C_{2M} = 0$; $C_{2M} \neq 0$, $C_{3M} \neq 0$, $C_{1M} = 0$. Consequently, the mathematical boundary conditions (2.31)–(2.33) are applied in case of $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} \neq 0$.

Conditions $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} = 0$. With regard to Equations (2.28), (2.29), (2.35), (6.4), (6.5), we get

$$
\begin{aligned}
\varepsilon_{nM} &= -\frac{p_n}{\zeta_M}\left[1 - c_{3M}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right],\\
\varepsilon_{\varphi M} = \varepsilon_{\theta M} &= -\frac{p_n}{\zeta_M}\left[1 - \left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right],\\
\varepsilon'_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} &= -\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M}\right) - x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M\, x_M^{c_{3M}-1}}\right)\right],\\
\varepsilon'_{n\theta M} = s_{44M}\,\sigma_{n\theta M} &= -\Theta\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M}\right) - x_n^{c_{3M}-1}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M\, x_M^{c_{3M}-1}}\right)\right],\\
\sigma_{nM} &= -\frac{p_n}{\zeta_M}\left\{c_{1M} - c_{2M} - \left[c_{3M}\left(c_{1M}+c_{2M}\right) - 2c_{2M}\right]\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right\},\\
\sigma_{\varphi M} = \sigma_{\theta M} &= -\frac{p_n}{\zeta_M}\left[c_{1M} - c_{2M} - \left(c_{1M} - c_{2M}c_{3M}\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right],\\
\sigma_{nB} &= -\frac{\rho_{1B}\,p_n}{\zeta_M} - \frac{1}{s_{44M}}\left[\rho_{1B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M}\right) + \rho_{1B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M}\right)\right]\\
&\quad + \frac{\rho_{2B}\,p_n}{\zeta_M x_M^{c_{3M}-1}} + \frac{1}{s_{44M}}\left[\rho_{2B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M x_M^{c_{3M}-1}}\right) + \rho_{2B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M x_M^{c_{3M}-1}}\right)\right],
\end{aligned}
\tag{6.7}
$$

where Θ, x_M, s_{44M}, c_{iM} ($i = 1,2,3$) are given by Equations (1.8), (1.9), (2.13), (2.18), respectively. The coefficients ζ_M, ρ_{iB}, $\rho_{iB}^{(\tau)}$ ($i = 1,2$; $\tau = \varphi, \nu$) have the forms

$$
\zeta_M = c_{1M} - c_{2M} - \left[c_{3M}\left(c_{1M}+c_{2M}\right) - 2c_{2M}\right]\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1},
$$

$$\rho_{1B} = (\vartheta_1 + \vartheta_2 + \vartheta_3)(c_{1M} - c_{2M}),$$
$$\rho_{2B} = \Big\{\vartheta_1 [(c_{1M} + c_{2M}) c_{3M} - 2 c_{2M}] + (\vartheta_2 + \vartheta_3)(c_{1M} - c_{2M} c_{3M})\Big\} x_M^{c_{3M}-1},$$
$$\rho_{1B}^{(\varphi)} = \vartheta_1 + \vartheta_2, \quad \rho_{1B}^{(\nu)} = \Theta(\vartheta_1 + \vartheta_3),$$
$$\rho_{2B}^{(\varphi)} = (\vartheta_1 + \vartheta_2) x_M^{c_{3M}-1}, \quad \rho_{2B}^{(\nu)} = \Theta(\vartheta_1 + \vartheta_3) x_M^{c_{3M}-1}, \tag{6.8}$$

where γ_{iM}, ϑ_j ($i = 1, \ldots, 4$; $j = 1,2,3$) are given by Equations (2.22), (2.32), respectively. The normal stress p_n is given by Equations (2.30) or (2.34) for $N = 1$ or $N + 1 = 1, 2, \ldots,$ respectively. With regard to Equation (6.7) and $\left[\varepsilon_{\varphi M}^{(1)}\right]_{x_n = x_M} = -p_n^{(1)} \rho_M$ [1]–[22], the coefficient ρ_M in Equation (2.30) for $N = 1$ is derived as

$$\rho_M = \frac{1}{\zeta_M}\left[1 - \left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1}\right]. \tag{6.9}$$

Conditions $C_{1M} \neq 0$, $C_{3M} \neq 0$, $C_{2M} = 0$. With regard to Equations (2.28), (2.29), (2.35), (6.4), (6.5), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta_M}\left[1 - c_{3M}\left(\frac{x_M}{x_n}\right)^3\right],$$
$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta_M}\left[1 - \left(\frac{x_M}{x_n}\right)^3\right],$$
$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = -\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M}\right) - \frac{1}{x_n^3}\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{\zeta_M}\right)\right],$$
$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = -\Theta\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M}\right) - \frac{1}{x_n^3}\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^3}{\zeta_M}\right)\right],$$
$$\sigma_{nM} = -\frac{p_n}{\zeta_M}\left[c_{1M} - c_{2M} + 2(c_{1M} + 2c_{2M})\left(\frac{x_M}{x_n}\right)^3\right],$$
$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta_M}\left[c_{1M} - c_{2M} - (c_{1M} + 2c_{2M})\left(\frac{x_M}{x_n}\right)^3\right],$$
$$\sigma_{nB} = -\frac{\rho_{1B}\,p_n}{\zeta_M} - \frac{1}{s_{44M}}\left[\rho_{1B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M}\right) + \rho_{1B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M}\right)\right]$$
$$+ \frac{\rho_{3B}\,p_n x_M^3}{\zeta_M} + \frac{1}{s_{44M}}\left[\rho_{3B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{\zeta_M}\right) + \rho_{3B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^3}{\zeta_M}\right)\right], \tag{6.10}$$

where Θ, x_M, s_{44M}, c_{iM}; ρ_{1B}, $\rho_{1B}^{(\tau)}$ ($i = 1,2,3$; $\tau = \varphi, \nu$) are given by Equations (1.8), (1.9), (2.13), (2.18); (6.8), respectively. The coefficients ζ_M, ρ_{3B}, $\rho_{3B}^{(\tau)}$ $(\tau = \varphi, \nu)$ have the forms

$$\zeta_M = c_{1M} - c_{2M} + 2\left(c_{1M} + 2c_{2M}\right)\left(\frac{x_M}{x_{IN}}\right)^3, \quad \zeta_{MB} = 3\left(c_{1M} + c_{2M}\right),$$
$$\rho_{3B} = \frac{\left(\vartheta_2 + \vartheta_3 - 2\vartheta_1\right)\left(c_{1M} + 2c_{2M}\right)}{x_M^3},$$
$$\rho_{3B}^{(\varphi)} = \frac{\vartheta_1 + \vartheta_2}{x_M^3}, \quad \rho_{3B}^{(\nu)} = \frac{\Theta\left(\vartheta_1 + \vartheta_3\right)}{x_M^3}, \tag{6.11}$$

where γ_{iM}, ϑ_j $(i = 1, \ldots, 4;\ j = 1,2,3)$ are given by Equations (2.22), (2.32), respectively. The normal stress p_n is given by Equations (2.30) or (2.34) for $N = 1$ or $N + 1 = 1, 2, \ldots$, respectively. With regard to Equation (6.10) and $\left[\varepsilon_{\varphi M}^{(1)}\right]_{x_n = x_M} = -p_n^{(1)}\rho_M$ [1]–[22], the coefficient ρ_M in Equation (2.30) for $N = 1$ is derived as

$$\rho_M = \frac{1}{\zeta_M}\left[1 - \left(\frac{x_M}{x_{IN}}\right)^{-3}\right]. \tag{6.12}$$

Conditions $C_{2M} \neq 0$, $C_{3M} \neq 0$, $C_{1M} = 0$. With regard to Equations (2.28), (2.29), (2.35), (6.4), (6.5), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta_M}\left[c_{3M}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} - 2\left(\frac{x_M}{x_n}\right)^3\right],$$
$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta_M}\left[\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} - \left(\frac{x_M}{x_n}\right)^3\right],$$
$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = -\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M\, x_M^{c_{3M}-1}}\right)x_n^{c_{3M}-1} - \frac{1}{x_n^3}\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{\zeta_M}\right)\right],$$
$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = -\Theta\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M\, x_M^{c_{3M}-1}}\right)x_n^{c_{3M}-1} - \frac{1}{x_n^3}\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^3}{\zeta_M}\right)\right],$$
$$\sigma_{nM} = -\frac{p_n}{\zeta_M}\left\{\left[c_{3M}\left(c_{1M} + c_{2M}\right) - 2c_{2M}\right]\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} - 2\left(c_{1M} + 2c_{2M}\right)\left(\frac{x_M}{x_n}\right)^3\right\},$$
$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta_M}\left[\left(c_{1M} - c_{2M}c_{3M}\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} + \left(c_{1M} + 2c_{2M}\right)\left(\frac{x_M}{x_n}\right)^3\right],$$
$$\sigma_{nB} = -\frac{\rho_{1B}\, p_n}{\zeta_M}\left(\frac{1}{2} + \ln x_M\right) + \frac{\rho_{3B}\, p_n\, x_M^{c_{3M}}}{\zeta_M}$$

$$
-\frac{1}{s_{44M}}\left\{\rho_{1B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta_M}\left(\frac{1}{2}+\ln x_M\right)\right]+\rho_{3B}^{(\nu)}\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta_M}\left(\frac{1}{2}+\ln x_M\right)\right]\right\}
+\frac{1}{s_{44M}}\left[\rho_{3B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^{c_{3M}}}{\zeta_M}\right)+\rho_{3B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n x_M^{c_{3M}}}{\zeta_M}\right)\right],\quad Q=M,MB, \tag{6.13}
$$

where Θ, x_M, s_{44M}, c_{iM}; ρ_{jB}, $\rho_{jB}^{(\tau)}$ $(i=1,2,3;\ j=1,3;\ \tau=\varphi,\nu)$ are given by Equations (1.8), (1.9), (2.13), (2.18); (6.9), (6.12), respectively. The coefficient ζ_M has the form

$$
\zeta_M=\left\{\left[c_{3M}\left(c_{1M}+c_{2M}\right)-2\,c_{2M}\right]\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}+2}+2\left(c_{1M}+2c_{2M}\right)\right\}\left(\frac{x_M}{x_{IN}}\right)^3. \tag{6.14}
$$

The normal stress p_n is given by Equations (2.30) or (2.34) for $N=1$ or $N+1=1,2,\ldots$, respectively. With regard to Equation (6.13) and $\left[\varepsilon_{\varphi M}^{(1)}\right]_{x_n=x_M}=-p_n^{(1)}\rho_M$ [1]–[22], the coefficient ρ_M in Equation (2.30) for $N=1$ is derived as

$$
\rho_M=\frac{1}{\zeta_M}\left[\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1}-\left(\frac{x_M}{x_{IN}}\right)^3\right]. \tag{6.15}
$$

Conditions $C_{1M}\neq 0$, $C_{2M}\neq 0$, $C_{3M}\neq 0$. Finally, with regard to Equations (2.21), (2.27), (2.31)–(2.33), (6.4), (6.5), we get

$$
\begin{aligned}
\varepsilon_{nM}&=-\frac{1}{\zeta_M}\left(\zeta_{1M}+\zeta_{2M}\,c_{3M}\,x_n^{c_{3M}-1}-\frac{2\,\zeta_{3M}}{x_n^3}\right),\\
\varepsilon_{\varphi M}=\varepsilon_{\theta M}&=-\frac{1}{\zeta_M}\left(\zeta_{1M}+\zeta_{2M}\,x_n^{c_{3M}-1}+\frac{\zeta_{3M}}{x_n^3}\right),\\
\varepsilon_{n\varphi M}=s_{44M}\,\sigma_{n\varphi M}&=\frac{\partial}{\partial\varphi}\left(\frac{\zeta_{1M}}{\zeta_M}\right)+x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left(\frac{\zeta_{2M}}{\zeta_M}\right)+\frac{1}{x_n^3}\frac{\partial}{\partial\varphi}\left(\frac{\zeta_{1M}}{\zeta_M}\right),\\
\varepsilon_{n\theta M}=s_{44M}\,\sigma_{n\theta M}&=\Theta\left[\frac{\partial}{\partial\nu}\left(\frac{\zeta_{1M}}{\zeta_M}\right)+x_n^{c_{3M}-1}\frac{\partial}{\partial\nu}\left(\frac{\zeta_{2M}}{\zeta_M}\right)+\frac{1}{x_n^3}\frac{\partial}{\partial\nu}\left(\frac{\zeta_{3M}}{\zeta_M}\right)\right],\\
\sigma_{nM}&=-\frac{1}{\zeta_M}\left\{\zeta_{1M}\left(c_{1M}-c_{2M}\right)+\zeta_{2M}\left[\left(c_{1M}+c_{2M}\right)c_{3M}-2\,c_{2M}\right]x_n^{c_{3M}-1}\right.\\
&\qquad\left.-\frac{2\,\zeta_{3M}\left(c_{1M}+2\,c_{2M}\right)}{x_n^3}\right\},\\
\sigma_{\varphi M}=\sigma_{\theta M}&=-\frac{1}{\zeta_M}\left[\zeta_{1M}\left(c_{1M}-c_{2M}\right)+\zeta_{2M}\left(c_{1M}-c_{2M}\,c_{3M}\right)x_n^{c_{3M}-1}\right.
\end{aligned}
$$

$$+\frac{\zeta_{3M}\left(c_{1M}+2c_{2M}\right)}{x_n^3}\Bigg],$$

$$\sigma_{1M}=\eta_{1M}+\eta_{2M}\,x_n^{c_{3M}-1}+\frac{\eta_{3M}}{x_n^3},$$

$$w_M=\kappa_{1M}+\kappa_{2M}\,x_n^{2(c_{3M}-1)}+\frac{\kappa_{3M}}{x_n^6}+\kappa_{4M}\,x_n^{c_{3M}-1}+\frac{\kappa_{5M}}{x_n^3}+\kappa_{6M}\,x_n^{c_{3M}-4},$$

$$W_M=4\int\limits_0^{\pi/2}\int\limits_0^{\pi/2}\left[\frac{\kappa_{1M}}{3}\left(x_M^3-x_{IN}^3\right)+\frac{\kappa_{2M}}{2c_{3M}+1}\left(x_M^{2c_{3M}+1}-x_{IN}^{2c_{3M}+1}\right)\right.$$
$$+\frac{\kappa_{3M}}{3}\left(\frac{1}{x_{IN}^3}-\frac{1}{x_M^3}\right)+\frac{\kappa_{4M}}{c_{3M}+2}\left(x_M^{c_{3M}+2}-x_{IN}^{c_{3M}+2}\right)$$
$$\left.+\kappa_{5M}\ln\left(\frac{x_M}{x_{IN}}\right)+\frac{\kappa_{6M}}{c_{3M}-1}\left(x_M^{c_{3M}-1}-x_{IN}^{c_{3M}-1}\right)\right]d\varphi\,d\nu,$$

$$\sigma_{nB}=-\frac{\rho_{1B}\,\zeta_{1M}}{\zeta_M}-\frac{1}{s_{44M}}\left[\rho_{1B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{\zeta_{1M}}{\zeta_M}\right)+\rho_{1B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{\zeta_{1M}}{\zeta_M}\right)\right]$$
$$+\frac{\rho_{2B}\,\zeta_{2M}}{\zeta_M}+\frac{1}{s_{44M}}\left[\rho_{2B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{\zeta_{2M}}{\zeta_M}\right)+\rho_{2B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{\zeta_{2M}}{\zeta_M}\right)\right],\tag{6.16}$$

where Θ; x_{IN}; x_M; s_{44M}; c_{iM}; ρ_{iB}, $\rho_{iB}^{(\tau)}$ ($i=1,2,3$; $\tau=\varphi,\nu$) are given by Equations (1.8); (1.9); (2.13); (2.18); (6.8), (6.11), respectively. The coefficients ζ_{iM}, ζ_M, η_{iM}, κ_{jM} ($i=1,2,3$; $j=2\ldots,6$; Equation (6.6)) have the forms

$$\zeta_{iM}=-p_n\,\zeta_{i1M}+\sigma_{nB}\,\zeta_{i2M},\quad i=1,2,3,$$

$$\zeta_{11M}=\left(c_{1M}+c_{2M}\right)\left(2+c_{3M}\right)x_M^{c_{3M}-3},$$

$$\zeta_{12M}=\frac{\left[\left(c_{1M}+c_{2M}\right)c_{3M}-2c_{2M}\right]x_{IN}^{c_{3M}-1}}{x_M^2}+\frac{2\left(c_{1M}+2c_{2M}\right)x_M^{c_{3M}}}{x_{IN}^3},$$

$$\zeta_{21M}=-\frac{3\left(c_{1M}+c_{2M}\right)}{x_M^2},\quad \zeta_{22M}=\frac{c_{2M}-c_{1M}}{x_M^2}-\frac{2\left(c_{1M}+2c_{2M}\right)x_M}{x_{IN}^3},$$

$$\zeta_{31M}=\left(c_{1M}+c_{2M}\right)\left(1-c_{3M}\right)x_M^3,$$

$$\zeta_{32M}=x_M\left\{\left(c_{1M}-c_{2M}\right)x_M^{c_{3M}-1}-\left[\left(c_{1M}+c_{2M}\right)c_{3M}-2c_{2M}\right]x_{IN}^{c_{3M}-1}\right\},$$

$$\zeta_M=\left(c_{1M}+c_{2M}\right)$$
$$\times\left\{\left(c_{1M}-c_{2M}\right)\left(2+c_{3M}\right)x_M^{c_{3M}-3}-\frac{3\left[\left(c_{1M}+c_{2M}\right)c_{3M}-2c_{2M}\right]x_{IN}^{c_{3M}-1}}{x_M^2}\right.$$
$$\left.-\frac{2\left(c_{1M}+2c_{2M}\right)\left(1-c_{3M}\right)x_M^{c_{3M}}}{x_{IN}^3}\right\},$$

$$\eta_{1M}=-\frac{\zeta_{1M}\left(\gamma_{1M}+\gamma_{2M}\right)}{\zeta_M}-\frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{\zeta_{1M}}{\zeta_M}\right)+\gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{\zeta_{1M}}{\zeta_M}\right)\right],$$

$$
\begin{aligned}
\eta_{2M} = &- \frac{\zeta_{2M} \left(\gamma_{1M} c_{3M} + \gamma_{2M}\right)}{\zeta_M} \\
&- \frac{1}{s_{44M}} \left[\gamma_{3M} \frac{\partial}{\partial \varphi} \left(\frac{\zeta_{2M}}{\zeta_M}\right) + \gamma_{4M} \frac{\partial}{\partial \nu} \left(\frac{\zeta_{2M}}{\zeta_M}\right)\right], \\
\eta_{3M} = &- \frac{\zeta_{3M} \left(\gamma_{2M} - 2\gamma_{1M}\right)}{\zeta_M} - \frac{1}{s_{44M}} \left[\gamma_{3M} \frac{\partial}{\partial \varphi} \left(\frac{\zeta_{3M}}{\zeta_M}\right) + \gamma_{4M} \frac{\partial}{\partial \nu} \left(\frac{\zeta_{3M}}{\zeta_M}\right)\right], \\
\kappa_{1M} = &\frac{3 \left(c_{1M} - c_{2M}\right)}{2} \left(\frac{\zeta_{1M}}{\zeta_M}\right)^2 \\
&+ \frac{1}{s_{44M}} \left\{\left[\frac{\partial}{\partial \varphi} \left(\frac{\zeta_{1M}}{\zeta_M}\right)\right]^2 + \Theta^2 \left[\frac{\partial}{\partial \nu} \left(\frac{\zeta_{1M}}{\zeta_M}\right)\right]^2\right\}, \\
\kappa_{2M} = &\left[\frac{\left(c_{1M} + c_{2M}\right) c_{3M}^2}{2} + c_{1M} - 2 c_{2M} c_{3M}\right] \left(\frac{\zeta_{2M}}{\zeta_M}\right)^2 \\
&+ \frac{1}{s_{44M}} \left\{\left[\frac{\partial}{\partial \varphi} \left(\frac{\zeta_{1M}}{\zeta_M}\right)\right]^2 + \Theta^2 \Bigg(\frac{\partial}{\partial \nu} \left(\frac{\zeta_{1M}}{\zeta_M}\right)\Bigg]^2\right\}, \\
\kappa_{3M} = &3 \left(c_{1M} + 2 c_{2M}\right) \left(\frac{\zeta_{3M}}{\zeta_M}\right)^2 \\
&+ \frac{1}{s_{44M}} \left\{\left[\frac{\partial}{\partial \varphi} \left(\frac{\zeta_{3M}}{\zeta_M}\right)\right]^2 + \Theta^2 \left[\frac{\partial}{\partial \nu} \left(\frac{\zeta_{3M}}{\zeta_M}\right)\right]^2\right\}, \\
\kappa_{4M} = &\zeta_{1M} \zeta_{2M} \left(c_{1M} - c_{2M}\right) \left(2 + c_{3M}\right) \left(\frac{1}{\zeta_M}\right)^2 \\
&+ \frac{2}{s_{44M}} \left[\frac{\partial}{\partial \varphi} \left(\frac{\zeta_{1M}}{\zeta_M}\right) \frac{\partial}{\partial \varphi} \left(\frac{\zeta_{2M}}{\zeta_M}\right) + \Theta^2 \frac{\partial}{\partial \nu} \left(\frac{\zeta_{1M}}{\zeta_M}\right) \frac{\partial}{\partial \nu} \left(\frac{\zeta_{2M}}{\zeta_M}\right)\right], \\
\kappa_{5M} = &\frac{2}{s_{44M}} \left[\frac{\partial}{\partial \varphi} \left(\frac{\zeta_{1M}}{\zeta_M}\right) \frac{\partial}{\partial \varphi} \left(\frac{\zeta_{3M}}{\zeta_M}\right) + \Theta^2 \frac{\partial}{\partial \nu} \left(\frac{\zeta_{1M}}{\zeta_M}\right) \frac{\partial}{\partial \nu} \left(\frac{\zeta_{3M}}{\zeta_M}\right)\right], \\
\kappa_{6M} = &\zeta_{2M} \zeta_{3M} \left[2 c_{2M} \left(1 - c_{3M}\right) - c_{1M}\right] \left(\frac{1}{\zeta_M}\right)^2 \\
&+ \frac{2}{s_{44M}} \left[\frac{\partial}{\partial \varphi} \left(\frac{\zeta_{2M}}{\zeta_M}\right) \frac{\partial}{\partial \varphi} \left(\frac{\zeta_{3M}}{\zeta_M}\right) + \Theta^2 \frac{\partial}{\partial \nu} \left(\frac{\zeta_{2M}}{\zeta_M}\right) \frac{\partial}{\partial \nu} \left(\frac{\zeta_{3M}}{\zeta_M}\right)\right], \quad (6.17)
\end{aligned}
$$

where γ_{iM} $(i = 1, \ldots, 4)$ is given by Equation (2.22). The normal stress p_n is given by Equations (2.30) or (2.34) for $N = 1$ or $N + 1 = 1, 2, \ldots$, respectively. With regard to Equation (6.16) and $\left[\varepsilon_{\varphi M}\right]_{x_n = x_M} = -p_n \phi_M + \sigma_{nB} \phi_B$, the coefficients ϕ_M, ϕ_B in Equation (2.34) for $N + 1 = 1, 2, \ldots$ are derived as

$$
\phi_M = \frac{1}{\zeta_M} \left(\zeta_{11M} + \zeta_{21M} x_{IN}^{c_{3M} - 1} + \frac{\zeta_{31M}}{x_{IN}^3}\right),
$$

$$\phi_B = \frac{1}{\zeta_M}\left(\zeta_{12M} + \zeta_{22M}\, x_{IN}^{c_{3M}-1} + \frac{\zeta_{32M}}{x_{IN}^3}\right). \tag{6.18}$$

6.3 Ellipsoidal Inclusion

In case of the ellipsoidal inclusion, we get $C_{2IN} = C_{3IN} = 0$, otherwise we get $(u_{nIN})_{x_n \to 0} \longrightarrow \pm\,\infty$, $(\varepsilon_{IN})_{x_n \longrightarrow 0} \longrightarrow \pm\,\infty$, $(\sigma_{IN})_{r \to 0} \longrightarrow \pm\infty$ due to $c_3 < 0$ (see Equations (2.18), (6.4), (6.5)). With regard to Equations (2.37), (2.38), (6.4), (6.5), we get [1]–[22]

$$\begin{aligned}
&\varepsilon_{nM} = \varepsilon_{\varphi M} = \varepsilon_{\theta M} = -p_n\,\rho_M,\\
&\varepsilon_{n\varphi M} = s_{44IN}\,\sigma_{n\varphi M} = -\rho_M\,\frac{\partial p_n}{\partial\varphi},\\
&\varepsilon_{n\theta M} = s_{44IN}\,\sigma_{n\theta M} = -\Theta\,\rho_M\,\frac{\partial p_n}{\partial\nu},\\
&\sigma_{nM} = \sigma_{\varphi M} = \sigma_{\theta M} = -p_n,\\
&\sigma_{1M} = -\,\rho_M\left[p_n\left(\gamma_{1IN}+\gamma_{2IN}\right) + \frac{1}{s_{44IN}}\left(\gamma_{3IN}\,\frac{\partial p_n}{\partial\varphi} + \gamma_{4IN}\,\frac{\partial p_n}{\partial\nu}\right)\right],\\
&w_{IN} = \rho_M^2\left\{\frac{3p_n^2}{2\rho_M} + \frac{2}{s_{44IN}}\left[\left(\frac{\partial p_n}{\partial\varphi}\right)^2 + \left(\frac{\partial p_n}{\partial\nu}\right)^2\right]\right\},\\
&W_{IN} = \frac{4\rho_M^2}{3}\int_0^{\pi/2}\!\!\int_0^{\pi/2} x_{IN}^3\left\{\frac{3p_n^2}{2\rho_M} + \frac{2}{s_{44IN}}\left[\left(\frac{\partial p_n}{\partial\varphi}\right)^2 + \left(\frac{\partial p_n}{\partial\nu}\right)^2\right]\right\} d\varphi\, d\nu,
\end{aligned} \tag{6.19}$$

where Θ, s_{44IN}, γ_{iIN} $(i=1,\ldots,4)$ are given by Equations (1.8), (2.13), (2.22), respectively. The normal stress p_n is given by Equations (2.30) or (2.34) for $N=1$ or $N+1 = 1,2,\ldots$, respectively. With regard to Equation (6.19) and $\left(\varepsilon_{\varphi IN}\right)_{x_n = x_{IN}} = -p_n\,\rho_{IN}$ [1]–[22], the coefficient ρ_{IN} in Equations (2.30), (2.34) is derived as

$$\rho_{IN} = \frac{1-2\mu_{IN}}{E_{IN}}. \tag{6.20}$$

Chapter 7

Mathematical Model 5

7.1 Mathematical Procedure

Let the mathematical procedures ∂Eq. (2.24) $/\partial r$, Eq. (6.2) $/r$ be performed, and then we get

$$x_n \frac{\partial^2 U_n}{\partial x_n^2} + (1 - c_3) \frac{\partial U_n}{\partial x_n} = 0, \tag{7.1}$$

$$\frac{\partial U_n}{\partial x_n} = -s_{44} (c_1 + c_2) \left(x_n^2 \frac{\partial^3 u_n}{\partial x_n^3} + 4 x_n \frac{\partial^2 u_n}{\partial x_n^2} \right), \tag{7.2}$$

where s_{44} and c_1, c_2, $c_3 < 0$ are given by Equations (2.13) and (2.18), respectively. Let the mathematical procedure ∂Eq. (7.2) $/\partial r$ be performed, and then we get

$$\frac{\partial^2 U_n}{\partial x_n^2} = -s_{44} (c_1 + c_2) \left(x_n^2 \frac{\partial^4 u_n}{\partial x_n^4} + 6 x_n \frac{\partial^3 u_n}{\partial x_n^3} + 4 \frac{\partial^2 u_n}{\partial x_n^2} \right). \tag{7.3}$$

Let Equations (6.2), (6.3) be substituted to (7.1), and then we get

$$x_n^2 \frac{\partial^4 u_n}{\partial x_n^4} + (7 - c_3) x_n \frac{\partial^3 u_n}{\partial x_n^3} + 4 (2 - c_3) \frac{\partial^2 u_n}{\partial x_n^2} = 0. \tag{7.4}$$

Let u_n be assumed in the form $u_n = x_n^{\lambda}$, then we get

$$u_n = C_1 x_n + C_2 x_n^{c_3} + \frac{C_3}{x_n^2} + C_4, \tag{7.5}$$

where $C_1 \ldots, C_4$ are integration constants, which are determined by the mathematical boundary conditions in Section 2.3. With regard to Equations (2.1)–(2.4), (2.14)–(2.17), (2.26), (7.6), we get

$$\varepsilon_n = C_1 + C_2 c_3 x_n^{c_3 - 1} - \frac{2 C_3}{x_n^3},$$

$$\varepsilon_\varphi = \varepsilon_\theta = C_1 + C_2 x_n^{c_3 - 1} + \frac{C_3}{x_n^3} + \frac{C_4}{x_n},$$

$$
\begin{aligned}
\varepsilon_{n\varphi} &= s_{44}\,\sigma_{n\varphi} = \frac{\partial C_1}{\partial \varphi} + x_n^{c_3-1}\,\frac{\partial C_2}{\partial \varphi} + \frac{1}{x_n^3}\,\frac{\partial C_3}{\partial \varphi} + \frac{1}{x_n}\,\frac{\partial C_4}{\partial \varphi},\\
\varepsilon_{n\theta} &= s_{44}\,\sigma_{n\theta} = \Theta\left[\frac{\partial C_1}{\partial \nu} + x_n^{c_3-1}\,\frac{\partial C_2}{\partial \nu} + \frac{1}{x_n^3}\,\frac{\partial C_3}{\partial \nu} + \frac{1}{x_n}\,\frac{\partial C_4}{\partial \nu}\right],\\
\sigma_n &= C_1\,(c_1 - c_2) + C_2\,[(c_1 + c_2)\,c_3 - 2\,c_2]\,x_n^{c_3-1} - \frac{2\,C_3\,(c_1 + 2\,c_2)}{x_n^3} - \frac{2\,c_2\,C_4}{x_n},\\
\sigma_\varphi &= \sigma_\theta = C_1\,(c_1 - c_2) + C_2\,(c_1 - c_2\,c_3)\,x_n^{c_3-1} + \frac{C_3\,(c_1 + 2\,c_2)}{x_n^3} + \frac{c_1\,C_4}{x_n},\\
\sigma_1 &= \eta_1 + \eta_2\,x_n^{c_3-1} + \frac{\eta_3}{x_n^3} + \frac{\eta_4}{x_n},\\
w &= \kappa_1 + \kappa_2\,x_n^{2(c_3-1)} + \frac{\kappa_3}{x_n^6} + \frac{\kappa_4}{x_n^2} + (\kappa_5 + \kappa_9)\,x_n^{c_3-1}\\
&\quad + \frac{\kappa_6}{x_n^3} + \kappa_7\,x_n^{c_3-4} + \frac{\kappa_8}{x_n} + \frac{\kappa_{10}}{x_n^4},
\end{aligned}
\tag{7.6}
$$

where Θ and η_i, κ_i $(i = 1,2,3)$ $(i = 1,2,3)$ are given by Equations (1.8) and (6.6), respectively. The coefficients η_4, κ_j $(i = 4,\ldots,6)$ are derived as

$$
\begin{aligned}
\eta_4 &= C_4\,\gamma_2 + \frac{1}{s_{44}}\left(\gamma_3\,\frac{\partial C_4}{\partial \varphi} + \gamma_4\,\frac{\partial C_4}{\partial \nu}\right),\\
\kappa_4 &= c_1\,C_4^2 + \frac{1}{s_{44}}\left[\left(\frac{\partial C_4}{\partial \varphi}\right)^2 + \Theta^2\left(\frac{\partial C_4}{\partial \nu}\right)^2\right],\\
\kappa_5 &= (c_1 - c_2)\,(2 + c_3)\,C_1\,C_2 + \frac{2}{s_{44}}\left(\frac{\partial C_1}{\partial \varphi}\,\frac{\partial C_2}{\partial \varphi} + \Theta^2\,\frac{\partial C_1}{\partial \nu}\,\frac{\partial C_2}{\partial \nu}\right),\\
\kappa_6 &= \frac{2}{s_{44}}\left(\frac{\partial C_1}{\partial \varphi}\,\frac{\partial C_3}{\partial \varphi} + \Theta^2\,\frac{\partial C_1}{\partial \nu}\,\frac{\partial C_3}{\partial \nu}\right),\\
\kappa_7 &= [2\,c_2\,(1 - c_3) - c_1]\,C_2\,C_3 + \frac{2}{s_{44}}\left(\frac{\partial C_2}{\partial \varphi}\,\frac{\partial C_3}{\partial \varphi} + \Theta^2\,\frac{\partial C_2}{\partial \nu}\,\frac{\partial C_3}{\partial \nu}\right),\\
\kappa_8 &= (c_1 - c_2)\,C_1\,C_4 + \frac{1}{s_{44}}\left(\frac{\partial C_1}{\partial \varphi}\,\frac{\partial C_4}{\partial \varphi} + \Theta^2\,\frac{\partial C_1}{\partial \nu}\,\frac{\partial C_4}{\partial \nu}\right),\\
\kappa_9 &= (c_1 - c_2\,c_3)\,C_2\,C_4 + \frac{1}{s_{44}}\left(\frac{\partial C_2}{\partial \varphi}\,\frac{\partial C_4}{\partial \varphi} + \Theta^2\,\frac{\partial C_2}{\partial \nu}\,\frac{\partial C_4}{\partial \nu}\right),\\
\kappa_{10} &= (c_1 + 2\,c_2)\,C_3\,C_4 + \frac{1}{s_{44}}\left(\frac{\partial C_3}{\partial \varphi}\,\frac{\partial C_4}{\partial \varphi} + \Theta^2\,\frac{\partial C_3}{\partial \nu}\,\frac{\partial C_4}{\partial \nu}\right),
\end{aligned}
\tag{7.7}
$$

where γ_i $(i = 1,\ldots,4)$ is given by Equation (2.22). In case of the ellipsoidal inclusion, we get $C_{2IN} = C_{3IN} = C_{4IN} = 0$, otherwise we get $(u_{nIN})_{x_n\to 0} \longrightarrow \pm\,\infty$, $(\varepsilon_{IN})_{x_n\longrightarrow 0} \longrightarrow \pm\,\infty$, $(\sigma_{IN})_{r\to 0} \longrightarrow \pm\infty$ due to $c_3 < 0$ (see Equations (2.18), (6.4)–

(6.10)). In case of $C_{1IN} \neq 0$ (see Equations (6.4), (7.5)), the mathematical solutions for the ellipsoidal inclusion is presented in Section 6.3.

7.2 Cell Matrix

The stress σ_{nB} is determined by two mathematical boundary conditions, i.e., by Equations (2.28), (2.29). With regard to Equations (6.4), (7.5), the condition $C_{iM} \neq 0$, $C_{4M} \neq 0$, $C_{jM} = C_{kM} = 0$ ($i, j, k = 1,2,3$; $i \neq j \neq k$) is considered in Section 7.2, where the condition $C_{iM} \neq 0$, $C_{jM} \neq 0$, $C_{kM} = 0$ is analysed in Section 6.2. Consequently, the mathematical boundary conditions (2.31)–(2.33) are applied in case of $C_{iM} \neq 0$, $C_{jM} \neq 0$, $C_{4M} \neq 0$, $C_{kM} = 0$ ($i, j, k = 1,2,3$; $i \neq j \neq k$), where the condition $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} \neq 0$ is analysed in Section 6.2.

Conditions $C_{1M} \neq 0$, $C_{4M} \neq 0$, $C_{2M} = C_{3M} = 0$. With regard to Equations (2.28), (2.29), (2.35), (7.5), (7.6), we get

$$
\begin{aligned}
\varepsilon_{nM} &= -\frac{p_n}{\zeta_M}, \\
\varepsilon_{\varphi M} &= \varepsilon_{\theta M} = -\frac{p_n}{\zeta_M}\left(1 - \frac{1}{x_n}\right), \\
\varepsilon_{n\varphi M} &= s_{44M}\,\sigma_{n\varphi M} = -\left(1 - \frac{1}{x_n}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M}\right), \\
\varepsilon_{n\theta M} &= s_{44M}\,\sigma_{n\theta M} = -\Theta\left(1 - \frac{1}{x_n}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M}\right), \\
\sigma_{nM} &= -\frac{p_n}{\zeta_M}\left(c_{1M} - c_{2M} + \frac{2c_{Mm}}{x_n}\right), \\
\sigma_{\varphi M} &= \sigma_{\theta M} = -\frac{p_n}{\zeta_M}\left(c_{1M} - c_{2M} - \frac{c_{1M}}{x_n}\right), \\
\sigma_{nB} &= -\frac{\rho_{1B}\,p_n}{\zeta_M} - \frac{1}{s_{44M}}\left[\rho_{1B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M}\right) + \rho_{1B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M}\right)\right] \\
&\quad + \frac{\rho_{4B}\,p_n}{\zeta_M} + \frac{1}{s_{44M}}\left[\rho_{4B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M}\right) + \rho_{4B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M}\right)\right], \qquad (7.8)
\end{aligned}
$$

where Θ; x_{IN}, x_M; s_{44M}; c_{iM}; ρ_{1B}, $\rho_{1B}^{(\tau)}$ ($i = 1,2$; $\tau = \varphi, \nu$) are given by Equations (1.8), (1.9), (2.13), (2.18); (6.8), respectively. The coefficients ζ_M, $\rho_{4B}^{(\tau)}$ ($\tau = \varphi, \nu$) have the forms

$$\zeta_M = c_{1M} - c_{2M} + \frac{2c_{2M}\,x_M}{x_{IN}},$$

$$\rho_{4B} = \frac{(\vartheta_2+\vartheta_3)\,c_{1M} - 2\,\vartheta_1\,c_{2M}}{x_M}, \quad \rho_{4B}^{(\varphi)} = \frac{\vartheta_1+\vartheta_2}{x_M}, \quad \rho_{4B}^{(\nu)} = \frac{\Theta\,(\vartheta_1+\vartheta_3)}{x_M}, \tag{7.9}$$

where ϑ_i $(i=1,2,3)$ is given by Equation (2.32). The normal stress p_n is given by Equations (2.30) or (2.34) for $N=1$ or $N+1=1,2,\ldots$, respectively. With regard to Equation (7.8) and $\left[\varepsilon_{\varphi M}^{(1)}\right]_{x_n=x_M} = -p_n^{(1)}\,\rho_M$ [1]–[22], the coefficient ρ_M in Equation (2.30) for $N=1$ is derived as

$$\rho_M = \frac{1}{\zeta_M}\left(1-\frac{1}{x_{IN}}\right). \tag{7.10}$$

Conditions $C_{2M}\neq 0$, $C_{4M}\neq 0$, $C_{1M}=C_{3M}=0$. With regard to Equations (2.28), (2.29), (2.35), (7.5), (7.6), we get

$$\varepsilon_{nM} = -\frac{p_n\,c_{3M}}{\zeta_M}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1},$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta_M}\left[\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} - \frac{1}{x_n}\right],$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = -\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M\,x_M^{c_{3M}-1}}\right) - \frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M}\right)\right],$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = -\Theta\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M\,x_M^{c_{3M}-1}}\right) - \frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M}\right)\right],$$

$$\sigma_{nM} = -\frac{p_n}{\zeta_M}\left\{\left[c_{3M}\,(c_{1M}+c_{2M}) - 2c_{2M}\right]\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} + \frac{2c_{2M}}{x_n}\right\},$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta_M}\left[(c_{1M}-c_{2M}c_{3M})\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} - \frac{c_{1M}}{x_n}\right],$$

$$\sigma_{nB} = -\frac{\rho_{2B}\,p_n}{\zeta_M\,x_M^{c_{3M}-1}} - \frac{1}{s_{44M}}\left[\rho_{2B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M\,x_M^{c_{3M}-1}}\right) + \rho_{2B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M\,x_M^{c_{3M}-1}}\right)\right]$$
$$+\frac{\rho_{4B}\,p_n}{\zeta_M} + \frac{1}{s_{44M}}\left[\rho_{4B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M}\right) + \rho_{4B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M}\right)\right], \tag{7.11}$$

where Θ; x_{IN}, x_M; s_{44M}; c_{iM}; ρ_{jB}, $\rho_{jB}^{(\tau)}$ ($i=1,2;\ j=2,4;\ \tau=\varphi,\nu$) are given by Equations (1.8); (1.9); (2.13); (2.18); (6.8), (7.9), respectively, and ζ_M has the form

$$\zeta_M = \left[c_{3M}\left(c_{1M}+c_{2M}\right)-2c_{2M}\right]\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1}+\frac{2c_{2M}\,x_M}{x_{IN}}. \tag{7.12}$$

The normal stress p_n is given by Equations (2.30) or (2.34) for $N=1$ or $N+1=1,2,\ldots$, respectively. With regard to Equation (7.11) and $\left[\varepsilon_{\varphi M}^{(1)}\right]_{x_n=x_M} = -p_n^{(1)}\rho_M$ [1]–[22], the coefficient ρ_M in Equation (2.30) for $N=1$ is derived as

$$\rho_M = \frac{1}{\zeta_M}\left[\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1}-\frac{1}{x_{IN}}\right], \quad Q=M,MB. \tag{7.13}$$

Conditions $C_{3M}\neq 0$, $C_{4M}\neq 0$, $C_{1M}=C_{2M}=0$. With regard to Equations (2.28), (2.29), (2.35), (7.5), (7.6), we get

$$\begin{aligned}
\varepsilon_{nM} &= \frac{2p_n}{\zeta_M}\left(\frac{x_{IN}}{x_n}\right)^3,\\
\varepsilon_{\varphi M} = \varepsilon_{\theta M} &= -\frac{p_n}{\zeta_M}\left[\left(\frac{x_{IN}}{x_n}\right)^3-\frac{1}{x_n}\right],\\
\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} &= -\left[\left(\frac{x_{IN}}{x_n}\right)^3-\frac{1}{x_n}\right]\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M}\right),\\
\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} &= -\left[\left(\frac{x_{IN}}{x_n}\right)^3-\frac{1}{x_n}\right]\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M}\right),\\
\sigma_{nM} &= \frac{2p_n}{\zeta_M}\left[\left(c_{1M}+2c_{2M}\right)\left(\frac{x_{IN}}{x_n}\right)^3-\frac{c_{2M}}{x_n}\right],\\
\sigma_{\varphi M} = \sigma_{\theta M} &= -\frac{p_n}{\zeta_M}\left[\left(c_{1M}-2c_{2M}\right)\left(\frac{x_{IN}}{x_n}\right)^3-\frac{c_{1M}}{x_n}\right],\\
\sigma_{nB} &= -\frac{\rho_{3B}\,p_n}{\zeta_M x_M^3}-\frac{1}{s_{44M}}\left[\rho_{3B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M x_M^3}\right)+\rho_{3B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M x_M^3}\right)\right]\\
&\quad+\frac{\rho_{4B}\,p_n}{\zeta_M}+\frac{1}{s_{44M}}\left[\rho_{4B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_M}\right)+\rho_{4B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_M}\right)\right],
\end{aligned} \tag{7.14}$$

where Θ; x_{IN}, x_M; s_{44M}; c_{iM}; ρ_{jB}, $\rho_{jB}^{(\tau)}$ ($i=1,2;\ j=3,4;\ \tau=\varphi,\nu$) are given by Equations (1.8); (1.9); (2.13); (2.18); (6.11), (7.9), respectively, and ζ_M has the form

$$\zeta_M = -\left[2\left(c_{1M}+2c_{2M}\right)+2c_{2M}\left(\frac{x_{IN}}{x_M}\right)^2\right]. \tag{7.15}$$

The normal stress p_n is given by Equations (2.30) or (2.34) for $N=1$ or $N+1=1,2,\ldots,$ respectively. With regard to Equation (7.14) and $\left[\varepsilon_{\varphi M}^{(1)}\right]_{x_n=x_M} = -p_n^{(1)}\rho_M$ [1]–[22], the coefficient ρ_M in Equation (2.30) for $N=1$ is derived as

$$\rho_M = \frac{x_{IN}-1}{\zeta_M\, x_{IN}}. \tag{7.16}$$

Conditions $C_{1M}\neq 0$, $C_{2M}\neq 0$, $C_{4M}\neq 0$, $C_{3M}=0$. Finally, with regard to Equations (2.21), (2.27), (2.31)–(2.33), (7.5), (7.6), we get

$$\begin{aligned}
&\varepsilon_{nM} = -\frac{p_n}{\zeta_M}\left(\zeta_{1M}+\zeta_{2M}\,c_{3M}\,x_n^{c_{3M}-1}\right),\\
&\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta_M}\left(\zeta_{1M}+\zeta_{2M}\,x_n^{c_{3M}-1}+\frac{\zeta_{4M}}{x_n}\right),\\
&\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = \frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_{1M}}{\zeta_M}\right)+x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_{2M}}{\zeta_M}\right)+\frac{1}{x_n}\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_{4M}}{\zeta_M}\right),\\
&\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} =\\
&\quad\Theta\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_{1M}}{\zeta_M}\right)+x_n^{c_{3M}-1}\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_{2M}}{\zeta_M}\right)+\frac{1}{x_n}\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_{4M}}{\zeta_M}\right)\right],\\
&\sigma_{nM} =\\
&\quad -\frac{p_n}{\zeta_M}\left\{\zeta_{1M}\left(c_{1M}-c_{2M}\right)+\zeta_{2M}\left[\left(c_{1M}+c_{2M}\right)c_{3M}-2\,c_{2M}\right]x_n^{c_{3M}-1}-\frac{2\,\zeta_{4M}\,c_{2M}}{x_n}\right\},\\
&\sigma_{\varphi M} = \sigma_{\theta M} =\\
&\quad -\frac{p_n}{\zeta_M}\left[\zeta_{1M}\left(c_{1M}-c_{2M}\right)+\zeta_{2M}\left(c_{1M}-c_{2M}\,c_{3M}\right)x_n^{c_{3M}-1}+\frac{\zeta_{4M}\,c_{1M}}{x_n}\right],\\
&\sigma_{1M} = \eta_{1M}+\eta_{2M}\,x_n^{c_{3M}-1}+\frac{\eta_{4M}}{x_n},\\
&w_M = \kappa_{1M}+\kappa_{2M}\,x_n^{2(c_{3M}-1)}+\frac{\kappa_{4M}}{x_n^2}+\left(\kappa_{5M}+\kappa_{9M}\right)x_n^{c_{3M}-1}+\frac{\kappa_{8M}}{x_n},\\
&W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\kappa_{1M}}{3}\left(x_M^3-x_{IN}^3\right)+\frac{\kappa_{2M}}{2\,c_{3M}+1}\left(x_M^{2\,c_{3M}+1}-x_{IN}^{2\,c_{3M}+1}\right)\right.\\
&\qquad\qquad +\kappa_{4M}\left(x_M-x_{IN}\right)+\frac{\kappa_{5M}+\kappa_{9M}}{c_{3M}+2}\left(x_M^{c_{3M}+2}-x_{IN}^{c_{3M}+2}\right)\\
&\qquad\qquad \left.+\frac{\kappa_{8M}}{2}\left(x_M^2-x_{IN}^2\right)\right]d\varphi\, d\nu,
\end{aligned}$$

$$\sigma_{nB} = -\frac{\rho_{1B}\,p_n\,\zeta_{1M}}{\zeta_M} - \frac{1}{s_{44M}}\left[\rho_{1B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{1M}}{\zeta_M}\right) + \rho_{1B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{1M}}{\zeta_M}\right)\right]$$
$$-\frac{\rho_{2B}\,p_n\,\zeta_{2M}}{\zeta_M} - \frac{1}{s_{44M}}\left[\rho_{2B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{2M}}{\zeta_M}\right) + \rho_{2B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{2M}}{\zeta_M}\right)\right]$$
$$-\frac{\rho_{4B}\,p_n\,\zeta_{4M}}{\zeta_M} - \frac{1}{s_{44M}}\left[\rho_{4B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{4M}}{\zeta_M}\right) + \rho_{4B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{4M}}{\zeta_M}\right)\right], \quad (7.17)$$

where Θ; x_{IN}, x_M; s_{44M}; c_{iM}; ρ_{jB}, $\rho_{jB}^{(\tau)}$ $(i=1,2,3;\ j=1,2,4;\ \tau=\varphi,\nu)$ are given by Equations (1.8); (1.9); (2.13); (2.18); (6.8), (7.9); respectively. The coefficients ζ_{iM}, ζ_M, η_{4M}, κ_{jM}, ρ_{4B}, $\rho_{4B}^{(\tau)}$ $(i=1,2,4;\ j=4,5,8,9;\ \tau=\varphi,\nu$; see Equation (7.7)) have the forms

$$\zeta_{iM} = -p_n\,\zeta_{i1M} + \sigma_{nB}\,\zeta_{i2M}, \quad i=1,2,4,$$
$$\zeta_{11M} = (c_{1M}+c_{2M})\,c_{3M}\,x_M^{c_{3M}-1},$$
$$\zeta_{12M} = [(c_{1M}+c_{2M})\,c_{3M} - 2c_{2M}]\,x_{IN}^{c_{3M}-1} + \frac{2\,c_{2M}\,x_M^{c_{3M}}}{x_{IN}},$$
$$\zeta_{21M} = -(c_{1M}+c_{2M}), \quad \zeta_{22M} = c_{2M}\left(1 - \frac{2x_M}{x_{IN}}\right) - c_{1M},$$
$$\zeta_{41M} = (c_{1M}+c_{2M})\,(1-c_{3M})\,x_M^{c_{3M}},$$
$$\zeta_{42M} = (c_{1M}-c_{2M})\,x_M^{c_{3M}} - [(c_{1M}+c_{2M})\,c_{3M} - 2c_{2M}]\,x_M\,x_{IN}^{c_{3M}-1},$$
$$\zeta_M = (c_{1M}+c_{2M})\left\{(c_{1M}-c_{2M})\,c_{3M}\,x_M^{c_{3M}-1} - [(c_{1M}+c_{2M})\,c_{3M} - 2c_{2M}]\,x_{IN}^{c_{3M}-1} - \frac{2\,c_{2M}\,(1-c_{3M})\,x_M^{c_{3M}}}{x_{IN}}\right\},$$
$$\eta_{4M} = -\frac{p_n\,\zeta_{4M}\,\gamma_{2M}}{\zeta_M} - \frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{4M}}{\zeta_M}\right) + \gamma_{4M}\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{4M}}{\zeta_M}\right)\right],$$
$$\kappa_{4M} = c_{1M}\left(\frac{p_n\,\zeta_{4M}}{\zeta_M}\right)^2 + \frac{1}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{4M}}{\zeta_M}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{4M}}{\zeta_M}\right)\right]^2\right\},$$
$$\kappa_{5M} = \zeta_{1M}\,\zeta_{2M}\,(c_{1M}-c_{2M})\,(2+c_{3M})\left(\frac{p_n}{\zeta_M}\right)^2$$
$$+ \frac{2}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{1M}}{\zeta_M}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{2M}}{\zeta_M}\right) + \Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{1M}}{\zeta_M}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{2M}}{\zeta_M}\right)\right],$$
$$\kappa_{8M} = \zeta_{1M}\,\zeta_{4M}\,(c_{1M}-c_{2M})\left(\frac{p_n}{\zeta_M}\right)^2$$
$$+ \frac{2}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{1M}}{\zeta_M}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{4M}}{\zeta_M}\right) + \Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{1M}}{\zeta_M}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{4M}}{\zeta_M}\right)\right],$$
$$\kappa_{9M} = \zeta_{2M}\,\zeta_{4M}\,(c_{1M}-c_{2M}\,c_{3M})\left(\frac{p_n}{\zeta_M}\right)^2$$

$$+\frac{2}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_{2M}}{\zeta_M}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_{4M}}{\zeta_M}\right)+\Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_{2M}}{\zeta_M}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_{4M}}{\zeta_M}\right)\right], \tag{7.18}$$

where γ_{iM}, ϑ_j ($i=2,3,4$; $j=1,2,3$) are given by Equations (2.22), (2.32), respectively. The coefficients η_{iM}, κ_{iM} ρ_{iB}, $\rho_{iB}^{(\tau)}$ ($i=1,2$; $\tau=\varphi,\nu$) are given by Equation (6.17), respectively, where ζ_{iM}, ζ_M in Equation (6.17) are given by Equation (7.18). The normal stress p_n is given by Equations (2.30) or (2.34) for $N=1$ or $N+1=1,2,\ldots$, respectively. With regard to Equation (7.17) and $\left[\varepsilon_{\varphi M}\right]_{x_n=x_M}=-p_n\phi_M+\sigma_{nB}\phi_B$, the coefficients ϕ_M, ϕ_B in Equation (2.34) for $N+1=1,2,\ldots$ are derived as

$$\phi_M=\frac{1}{\zeta_M}\left(\zeta_{11M}+\zeta_{21M}x_{IN}^{c_{3M}-1}+\frac{\zeta_{41M}}{x_{IN}}\right),$$
$$\phi_B=\frac{1}{\zeta_M}\left(\zeta_{12M}+\zeta_{22M}x_{IN}^{c_{3M}-1}+\frac{\zeta_{42M}}{x_{IN}}\right). \tag{7.19}$$

Conditions $C_{1M}\neq 0$, $C_{3M}\neq 0$, $C_{4M}\neq 0$, $C_{2M}=0$. Consequently, with regard to Equations (2.21), (2.27), (2.31)–(2.33), (7.5), (7.6), we get

$$\varepsilon_{nM}=-\frac{p_n}{\zeta_M}\left(\zeta_{1M}-\frac{2\zeta_{3M}}{x_n^3}\right),$$
$$\varepsilon_{\varphi M}=\varepsilon_{\theta M}=-\frac{p_n}{\zeta_M}\left(\zeta_{1M}+\frac{\zeta_{3M}}{x_n^3}+\frac{\zeta_{4M}}{x_n}\right),$$
$$\varepsilon_{n\varphi M}=s_{44M}\sigma_{n\varphi M}=\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_{1M}}{\zeta_M}\right)+\frac{1}{x_n^3}\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_{3M}}{\zeta_M}\right)+\frac{1}{x_n}\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_{4M}}{\zeta_M}\right),$$
$$\varepsilon_{n\theta M}=s_{44M}\sigma_{n\theta M}=\Theta\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_{1M}}{\zeta_M}\right)+\frac{1}{x_n^3}\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_{3M}}{\zeta_M}\right)+\frac{1}{x_n}\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_{4M}}{\zeta_M}\right)\right],$$
$$\sigma_{nM}=$$
$$-\frac{p_n}{\zeta_M}\left\{\zeta_{1M}\left(c_{1M}-c_{2M}\right)-\frac{2\zeta_{3M}\left(c_{1M}+2c_{2M}\right)}{x_n^3}-\frac{2\zeta_{4M}c_{2M}}{x_n}\right\},$$
$$\sigma_{\varphi M}=\sigma_{\theta M}=$$
$$-\frac{p_n}{\zeta_M}\left[\zeta_{1M}\left(c_{1M}-c_{2M}\right)+\frac{\zeta_{3M}\left(c_{1M}+2c_{2M}\right)}{x_n^3}+\frac{\zeta_{4M}c_{1M}}{x_n}\right],$$
$$\sigma_{1M}=\eta_{1M}+\frac{\eta_{3M}}{x_n^3}+\frac{\eta_{4M}}{x_n},$$
$$w_M=\kappa_{1M}+\frac{\kappa_{3M}}{x_n^6}+\frac{\kappa_{4M}}{x_n^2}+\frac{\kappa_{6M}}{x_n^3}+\frac{\kappa_{8M}}{x_n}+\frac{\kappa_{10M}}{x_n^4},$$
$$W_M=4\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\kappa_{1M}}{3}\left(x_M^3-x_{IN}^3\right)+\frac{\kappa_{3M}}{3}\left(\frac{1}{x_{IN}^3}-\frac{1}{x_M^3}\right)+\kappa_{4M}\left(x_M-x_{IN}\right)\right.$$

$$+ \kappa_{6M} \ln\left(\frac{x_M}{x_{IN}}\right) + \frac{\kappa_{8M}}{2}\left(x_M^2 - x_{IN}^2\right) + \kappa_{10M}\left(\frac{1}{x_{IN}} - \frac{1}{x_M}\right)\Bigg] d\varphi\, d\nu,$$

$$\begin{aligned}
\sigma_{nB} = &- \frac{\rho_{1B}\, p_n \zeta_{1M}}{\zeta_M} - \frac{1}{s_{44M}}\left[\rho_{1B}^{(\varphi)} \frac{\partial}{\partial\varphi}\left(\frac{p_n \zeta_{1M}}{\zeta_M}\right) + \rho_{1B}^{(\nu)} \frac{\partial}{\partial\nu}\left(\frac{p_n \zeta_{1M}}{\zeta_M}\right)\right] \\
&+ \frac{\rho_{3B}\, p_n \zeta_{3M}}{\zeta_M} + \frac{1}{s_{44M}}\left[\rho_{3B}^{(\varphi)} \frac{\partial}{\partial\varphi}\left(\frac{p_n \zeta_{3M}}{\zeta_M}\right) + \rho_{3B}^{(\nu)} \frac{\partial}{\partial\nu}\left(\frac{p_n \zeta_{3M}}{\zeta_M}\right)\right] \\
&+ \frac{\rho_{4B}\, p_n \zeta_{4M}}{\zeta_M} + \frac{1}{s_{44M}}\left[\rho_{4B}^{(\varphi)} \frac{\partial}{\partial\varphi}\left(\frac{p_n \zeta_{4M}}{\zeta_M}\right) + \rho_{4B}^{(\nu)} \frac{\partial}{\partial\nu}\left(\frac{p_n \zeta_{4M}}{\zeta_M}\right)\right],
\end{aligned} \tag{7.20}$$

where Θ; x_{IN}, x_M; s_{44M}; c_{iM}; ρ_{jB}, $\rho_{jB}^{(\tau)}$ ($i=1,2,3$; $j=1,3,4$; $\tau = \varphi, \nu$) are given by Equations (1.8); (1.9); (2.13); (2.18); (6.8), (7.9); respectively. The coefficients ζ_{iM}, ζ_M, η_{4M}, κ_{jM}, ρ_{4B}, $\rho_{4B}^{(\tau)}$ ($i=1,3,4$; $j=4,5,8,9$; $k=1,2$; $\tau = \varphi, \nu$; see Equation (7.7)) have the forms

$$\begin{aligned}
&\zeta_{iM} = -p_n \zeta_{i1M} + \sigma_{nB} \zeta_{i2M}, \quad i = 1,3,4, \\
&\zeta_{11M} = -\frac{2\left(c_{1M} + c_{2M}\right)}{x_M^3}, \quad \zeta_{12M} = \frac{2}{x_{IN}}\left(\frac{c_{2M}}{x_M^2} - \frac{c_{1M} + 2c_{2M}}{x_{IN}^2}\right), \\
&\zeta_{31M} = -\left(c_{1M} + c_{2M}\right), \quad \zeta_{32M} = c_{2M}\left(1 - \frac{2x_M}{x_{IN}}\right) - c_{1M}, \\
&\zeta_{41M} = \frac{3\left(c_{1M} + c_{2M}\right)}{x_M^2}, \quad \zeta_{42M} = \left[\frac{c_{1M} - c_{2M}}{x_M^3} + \frac{2\left(c_{1M} + 2c_{2M}\right)}{x_{IN}^3}\right] x_M \\
&\zeta_M = -2\left(c_{1M} + c_{2M}\right)\left(\frac{c_{1M} - c_{2M}}{x_M^3} + \frac{c_{1M} + 2c_{2M}}{x_{IN}^3} + \frac{3c_{2M}}{x_{IN} x_M^2}\right), \\
&\kappa_{6M} = \frac{2}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n \zeta_{1M}}{\zeta_M}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n \zeta_{3M}}{\zeta_M}\right) + \Theta^2 \frac{\partial}{\partial\nu}\left(\frac{p_n \zeta_{1M}}{\zeta_M}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n \zeta_{3M}}{\zeta_M}\right)\right], \\
&\kappa_{10M} = \zeta_{3M} \zeta_{4M}\left(c_{1M} + 2c_{2M}\right)\left(\frac{p_n}{\zeta_M}\right)^2 \\
&\qquad + \frac{1}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n \zeta_{3M}}{\zeta_M}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n \zeta_{4M}}{\zeta_M}\right) + \Theta^2 \frac{\partial}{\partial\nu}\left(\frac{p_n \zeta_{3M}}{\zeta_M}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n \zeta_{4M}}{\zeta_M}\right)\right].
\end{aligned} \tag{7.21}$$

The coefficients η_{iM}, κ_{iM} ($i=1,3$) and η_{4M}, κ_{4M}, κ_{8M} are given by Equations (6.17) and (7.18), respectively, where ζ_{jM} ($j=1,3,4$), ζ_M in Equations (6.17), (7.18) are given by Equation (7.21). The normal stress p_n is given by Equations (2.30) or (2.34) for $N = 1$ or $N + 1 = 1, 2, \ldots$, respectively. With regard to Equation (7.20) and $\left[\varepsilon_{\varphi M}\right]_{x_n = x_M} = -p_n \phi_M + \sigma_{nB} \phi_B$, the coefficients ϕ_M, ϕ_B in Equation (2.34) for $N + 1 = 1, 2, \ldots$ are derived as

$$\phi_M = \frac{1}{\zeta_M}\left(\zeta_{11M} + \frac{\zeta_{31M}}{x_{IN}^3} + \frac{\zeta_{41M}}{x_{IN}}\right),$$

$$\phi_B = \frac{1}{\zeta_M}\left(\zeta_{12M} + \frac{\zeta_{32M}}{x_{IN}^3} + \frac{\zeta_{42M}}{x_{IN}}\right). \tag{7.22}$$

Conditions $C_{2M} \neq 0$, $C_{3M} \neq 0$, $C_{4M} \neq 0$, $C_{1M} = 0$. Finally, with regard to Equations (2.21), (2.27), (2.31)–(2.33), (7.5), (7.6), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta_M}\left(\zeta_{2M}\, c_{3M}\, x_n^{c_{3M}-1} - \frac{2\,\zeta_{3M}}{x_n^3}\right),$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = -\frac{p_n}{\zeta_M}\left(\zeta_{2M}\, x_n^{c_{3M}-1} + \frac{\zeta_{3M}}{x_n^3} + \frac{\zeta_{4M}}{x_n}\right),$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{2M}}{\zeta_M}\right) + \frac{1}{x_n^3}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{3M}}{\zeta_M}\right) + \frac{1}{x_n}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{4M}}{\zeta_M}\right),$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = -\,\Theta\left[x_n^{c_{3M}-1}\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{2M}}{\zeta_M}\right) + \frac{1}{x_n^3}\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{3M}}{\zeta_M}\right) + \frac{1}{x_n}\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{4M}}{\zeta_M}\right)\right],$$

$$\sigma_{nM} = -\frac{p_n}{\zeta_M}\left\{\zeta_{2M}\left[(c_{1M}+c_{2M})\,c_{3M} - 2\,c_{2M}\right]x_n^{c_{3M}-1} - \frac{2\,\zeta_{3M}\,(c_{1M}+2\,c_{2M})}{x_n^3} - \frac{2\,\zeta_{4M}\,c_{2M}}{x_n}\right\},$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta_M}\left[\zeta_{2M}\,(c_{1M} - c_{2M}\,c_{3M})\,x_n^{c_{3M}-1} + \frac{\zeta_{3M}\,(c_{1M}+2\,c_{2M})}{x_n^3} + \frac{\zeta_{4M}\,c_{1M}}{x_n}\right],$$

$$\sigma_{1M} = \eta_{2M}\,x_n^{c_{3M}-1} + \frac{\eta_{3M}}{x_n^3} + \frac{\eta_{4M}}{x_n},$$

$$w_M = \kappa_{2M}\,x_n^{2(c_{3M}-1)} + \frac{\kappa_{3M}}{x_n^6} + \frac{\kappa_{4M}}{x_n^2} + \kappa_{7M}\,x_n^{c_{3M}-4} + \kappa_{9M}\,x_n^{c_{3M}-1} + \frac{\kappa_{10M}}{x_n^4},$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\kappa_{2M}}{2\,c_{3M}+1}\left(x_M^{2\,c_{3M}+1} - x_{IN}^{2\,c_{3M}+1}\right) + \frac{\kappa_{3M}}{3}\left(\frac{1}{x_{IN}^3} - \frac{1}{x_M^3}\right) + \kappa_{4M}\,(x_M - x_{IN}) + \frac{\kappa_{7M}}{c_{3M}-1}\left(x_M^{c_{3M}-1} - x_{IN}^{c_{3M}-1}\right) + \frac{\kappa_{9M}}{c_{3M}+2}\left(x_M^{c_{3M}+2} - x_{IN}^{c_{3M}+2}\right) + \kappa_{10M}\left(\frac{1}{x_{IN}} - \frac{1}{x_M}\right)\right] d\varphi\, d\nu,$$

$$\sigma_{nB} = -\frac{\rho_{2B}\,p_n\,\zeta_{2M}}{\zeta_M} - \frac{1}{s_{44M}}\left[\rho_{2B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{2M}}{\zeta_M}\right) + \rho_{2B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{2M}}{\zeta_M}\right)\right] - \frac{\rho_{3B}\,p_n\,\zeta_{3M}}{\zeta_M} - \frac{1}{s_{44M}}\left[\rho_{3B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{3M}}{\zeta_M}\right) + \rho_{3B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{3M}}{\zeta_M}\right)\right]$$

$$-\frac{\rho_{4B}\,p_n\,\zeta_{4M}}{\zeta_M}-\frac{1}{s_{44M}}\left[\rho_{4B}^{(\varphi)}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{4M}}{\zeta_M}\right)+\rho_{4B}^{(\nu)}\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{4M}}{\zeta_M}\right)\right], \quad (7.23)$$

where Θ; x_{IN}, x_M; s_{44M}; c_{iM}; ρ_{jB}, $\rho_{jB}^{(\tau)}$ $(i=1,2,3;\ j=2,3,4;\ \tau=\varphi,\nu)$ are given by Equations (1.8); (1.9); (2.13); (2.18); (6.8), (7.9); respectively. The coefficients ζ_{iM} $(i=2,3,4)$, ζ_M, κ_{7M} have the forms

$$\begin{aligned}
&\zeta_{iM}=-p_n\,\zeta_{i1M}+\sigma_{nB}\,\zeta_{i2M}, \quad i=2,3,4,\\
&\zeta_{21M}=-\frac{2\,(c_{1M}+c_{2M})}{x_M^3}, \quad \zeta_{22M}=\frac{2}{x_{IN}}\left(\frac{c_{2M}}{x_M^2}-\frac{c_{1M}+2\,c_{2M}}{x_{IN}^2}\right),\\
&\zeta_{31M}=-\,(c_{1M}+c_{2M})\,c_{3M}\,x_M^{c_{3M}-1},\\
&\zeta_{32M}=-\left\{[(c_{1M}+c_{2M})\,c_{3M}-2\,c_{2M}]\,x_{IN}^{c_{3M}-1}+\frac{2\,c_{2M}\,x_M^{c_{3M}}}{x_{IN}}\right\},\\
&\zeta_{41M}=(c_{1M}+c_{2M})\,(2+c_{3M})\,x_M^{c_{3M}-3},\\
&\zeta_{42M}=[(c_{1M}+c_{2M})\,c_{3M}-2\,c_{2M}]\,\frac{x_{IN}^{c_{3M}-1}}{x_M^2}+\frac{2\,(c_{1M}+2\,c_{2M})\,x_M^{c_{3M}}}{x_{IN}^3},\\
&\zeta_M=2\,(c_{1M}+c_{2M})\left\{\frac{(c_{1M}+2\,c_{2M})\,c_{3M}\,x_M^{c_{3M}-1}}{x_{IN}^3}-\frac{(2+c_{3M})\,c_{2M}\,x_M^{c_{3M}-3}}{x_{IN}}\right.\\
&\qquad\left.-\frac{[(c_{1M}+c_{2M})\,c_{3M}-2\,c_{2M}]\,x_{IN}^{c_{3M}-1}}{x_M^3}\right\},\\
&\kappa_{7M}=\zeta_{2M}\,\zeta_{3M}\,[2\,c_{2M}\,(1-c_{3M})-c_{1M}]\left(\frac{p_n}{\zeta_M}\right)^2\\
&\quad+\frac{1}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{2M}}{\zeta_M}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\,\zeta_{3M}}{\zeta_M}\right)\Theta^2\,\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{2M}}{\zeta_M}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\,\zeta_{3M}}{\zeta_M}\right)\right]. \quad (7.24)
\end{aligned}$$

The coefficients η_{iM}, κ_{iM} $(i=2,3)$; η_{4M}, κ_{4M}, κ_{9M}; κ_{120M}; are given by Equations (6.17); (7.18); (7.21), respectively, where ζ_{jM} $(j=1,3,4)$, ζ_M in Equations (6.17), (7.18), (7.21) are given by Equation (7.24). The normal stress p_n is given by Equations (2.30) or (2.34) for $N=1$ or $N+1=1,2,\ldots$, respectively. With regard to Equation (7.23) and $\left[\varepsilon_{\varphi M}\right]_{x_n=x_M}=-p_n\,\phi_M+\sigma_{nB}\,\phi_B$, the coefficients ϕ_M, ϕ_B in Equation (2.34) for $N+1=1,2,\ldots$ are derived as

$$\begin{aligned}
\phi_M&=\frac{1}{\zeta_M}\left(\zeta_{21M}\,x_{IN}^{c_{3M}-1}+\frac{\zeta_{31M}}{x_{IN}^3}+\frac{\zeta_{41M}}{x_{IN}}\right),\\
\phi_B&=\frac{1}{\zeta_M}\left(\zeta_{22M}\,x_{IN}^{c_{3M}-1}+\frac{\zeta_{32M}}{x_{IN}^3}+\frac{\zeta_{42M}}{x_{IN}}\right). \quad (7.25)
\end{aligned}$$

Chapter 8

Strengthening

The analytical model of the micro-strengthening $\sigma_{st} = \sigma_{st}(x_1)$ and the macro-strengthening $\overline{\sigma_{st}}$ results from the following analysis [3, 4, 12, 13, 21]. Figures 8.1 and 8.2 shows the plane $x_2'x_3'$ in the cubic cell (see Figure 1.2) for $x_1 \in \langle 0, a_1 \rangle$ and $x_1 \in \langle a_1, d/2 \rangle$, respectively, where $[x_1, x_2, x_3]$ are coordinates of the point $P \subset x_2'x_3'$. The plane $O'P_1P_2$ with the ellipse E_{23} (see Figure 8.2) represents a cross section of the ellipsoid inclusion in the plane $x_2'x_3'$. With regard to Figures (8.1), (8.2), the goniometric functions in Equations (1.8), (1.9) have the forms

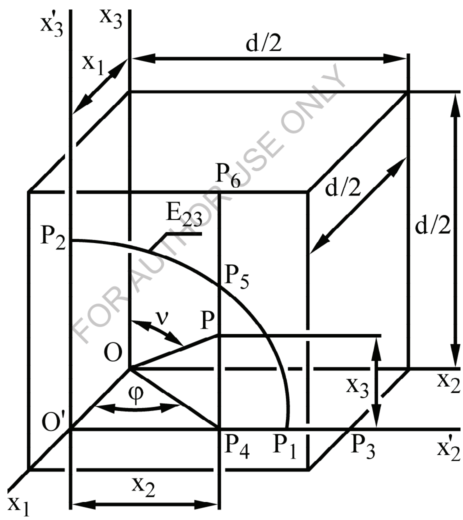


Figure 8.1: The plane $x_2'x_3'$ in the cubic cell (see Figure 1.2) for $x_1 \in \langle 0, a_1 \rangle$, where $[x_1, x_2, x_3]$ are coordinates of the point $P \subset x_2'x_3'$. The plane $O'P_1P_2$ with the ellipse E_{23} represents a cross section of the ellipsoid inclusion in the plane $x_2'x_3'$ (see Figure 1.2).

$$\sin\varphi = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}, \quad \cos\varphi = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \quad \tan\varphi = \frac{1}{\cot} = \frac{x_2}{x_1},$$

$$\sin\nu = \sqrt{\frac{x_1^2+x_2^2}{x_1^2+x_2^2+x_3^2}}, \quad \cos\nu = \frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}}, \quad x_n = \frac{x_3}{\cos\theta}, \qquad (8.1)$$

where $\cos\theta$ is given by Equation (1.6). With regard to Equation (1.2), the parameters b_2, b_3 of the ellipse E_{23} along the axes x_2', x_3', respectively, are derived as (see Figure 8.1)

$$b_2 = O'P_1 = \frac{a_2\sqrt{a_1^2-x_1^2}}{a_1}, \quad b_3 = O'P_2 = \frac{a_3\sqrt{a_1^2-x_1^2}}{a_1}, \qquad (8.2)$$

and then we get

$$b_4 = P_4P_5 = \frac{a_3\sqrt{b_2^2-x_2^2}}{a_2}. \qquad (8.3)$$

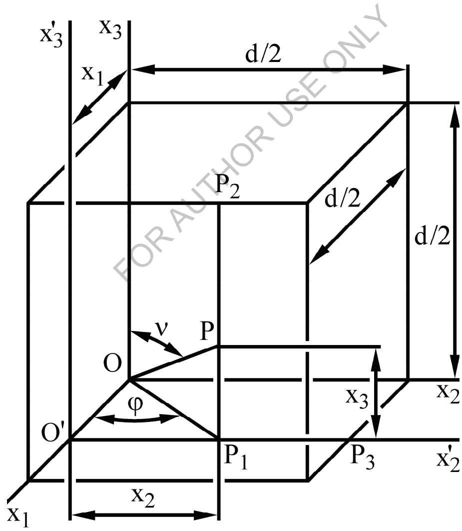


Figure 8.2: The plane $x_2'x_3'$ in the cubic cell (see Figure 1.2) for $x_1 \in \langle a_1, d/2\rangle$, where $[x_1,x_2,x_3]$ are coordinates of the point $P \subset x_2'x_3'$.

The micro-strengthening $\sigma_{st} = \sigma_{st}(x_1)$ represents a stress along the axis x_1, which is homogeneous at each point of the plane $x_2'x_3'$ with the area $S = d^2/4$, i.e., $\sigma_{st} \neq f(x_2,x_3)$.

If $x_1 \in \langle 0, a_1 \rangle$, then the elastic energy surface density W_{st}, which is induced by σ_{st} and accumulated within the area $S_{IN} = \pi b_2 b_3 / 4$ of the plane $O'P_1P_2$ and within the area $S_M = (d/2)^2 - S_{IN}$ of the plane $x_2' x_3'$ (see Figure 8.1), has the form

$$W_{st} = \omega \sigma_{1st}^2, \tag{8.4}$$

where σ_{1st} is related to $x_1 \in \langle 0, a_1 \rangle$. The coefficient ω is derived as

$$\omega = \frac{1}{8} \left[\pi b_2 b_3 \left(\frac{1}{E_{IN}} - \frac{1}{E_M} \right) + \frac{d^2}{E_M} \right], \tag{8.5}$$

where E_{IN} and E_M is Young's modulus for the ellipsoidal inclusion and the matrix, respectively. The elastic energy surface density W_{1S}, which is induced by the stresses $\sigma_{1IN} = \sigma_{1IN}(x_1)$ (see Equations (3.12), (6.19)) and $\sigma_{1M} = \sigma_{1M}(x_1)$ (see Equations (4.18), (5.17), (6.16), (7.17), (7.20), (7.23)), has the form

$$\begin{aligned}
W_{1S} &= \frac{1}{2} \left(\frac{W_{INS}}{E_{IN}} + \frac{W_{1MS}}{E_M} \right), \\
W_{INS} &= \int_0^{b_2} \left(\int_0^{b_4} \sigma_{1IN}^2 \, dx_3 \right) dx_2, \\
W_{1MS} &= \int_0^{b_2} \left(\int_{b_4}^{d/2} \sigma_{1M}^2 \, dx_3 \right) dx_2 + \int_{b_2}^{d/2} \left(\int_0^{d/2} \sigma_{1M}^2 \, dx_3 \right) x_2, \\
x_1 &\in \langle 0, a_1 \rangle .
\end{aligned} \tag{8.6}$$

The micro-strengthening $\sigma_{1st} = \sigma_{1st}(x_1)$ for $x_1 \in \langle 0, a_1 \rangle$, which results from the condition $W_{st} = W_{1S}$ [3, 4, 12, 13, 21], is derived as

$$\sigma_{1st} = \sqrt{\frac{1}{2\omega} \left(\frac{W_{INS}}{E_{IN}} + \frac{W_{1MS}}{E_M} \right)}, \quad x_1 \in \langle 0, a_1 \rangle . \tag{8.7}$$

If $x_1 \in \langle a_1, d/2 \rangle$, then the elastic energy surface density W_{st}, which is induced by σ_{st} and accumulated within the area $S_M = d^2/4$ of the plane $x_2' x_3'$ (see Figure 8.2), has the form

$$W_{st} = \frac{\sigma_{2st}^2 d^2}{8 E_M}, \tag{8.8}$$

where σ_{2st} is related to $x_1 \in \langle a_1, d/2 \rangle$. Similarly, we get

$$W_{2S} = \frac{W_{2MS}}{2E_M}, \quad W_{2MS} = \int_0^{d/2} \int_0^{d/2} \sigma_{1M}^2 \, dx_2 \, dx_3, \quad x_1 \in \left\langle a_1, \frac{d}{2} \right\rangle. \tag{8.9}$$

With regard to the condition $W_{st} = W_{2S}$ [3, 4, 12, 13, 21], we get

$$\sigma_{2st} = \frac{2\sqrt{W_{2S}}}{d}. \tag{8.10}$$

Finally, the macro-strengthening $\overline{\sigma_{st}}$ is derived as [3, 4, 12, 13, 21]

$$\overline{\sigma_{st}} = \frac{2}{d} \left(\int_0^{a_1} \sigma_{1st} \, dx_1 + \int_{a_1}^{d/2} \sigma_{2st} \, dx_1 \right). \tag{8.11}$$

If $\alpha_{IN} < \alpha_M$ or $\alpha_{IN} > \alpha_M$, the strengthening exhibits a resistive effect against compressive or tensile mechanical loading, respectively.

The macro-strengthening $\overline{\sigma_{st}} = \overline{\sigma_{st}}(v, a_1, a_2, a_3)$ is a function of the inclusion volume fraction v_{IN} and the dimensions a_1, a_2, a_3 of the ellipsoidal inclusion. In case of a real inclusion-matrix composite, such values of the microstructural parameters v_{IN}, a_1, a_2, a_3 can be numerically determined to result in a maximum value of $|\overline{\sigma_{st}}|$.

Chapter 9

Crack Formation

The analytical model of the crack formation in the matrix results from the following analysis [3, 4, 5, 19]–[22]. Figures 9.1, 9.3 show the ellipse E_{123} in the plane $x_{12}x_3$ of the cubic cell (see Figures (1.4), (1.5)), where $a_{12} = O4$, $x_{122} = O5$ are given by Equations (1.9), and $a_3 = O3$.

With regard to the plane $x_{12}x_3$ for $\varphi \in \langle 0, \pi/2 \rangle$ (see Figures 1.4, 1.5), the elastic energy density $w_Q = w_Q(x_n, \varphi, \nu)$ ($Q = IN, INB, M, MB$; see Equations (3.10), (3.14), (4.10), (4.13), (4.16), (4.19), (5.9), (5.12), (5.15), (5.18), (6.8), (6.11), (6.14), (6.17), (**??**), (7.9), (7.12), (7.15), (7.18), (7.21), (7.24)) is determined as a function of the coordinates x_n, $\nu \in \langle 0, \pi/2 \rangle$.

The elastic energy density $w_Q = w_Q(x_{12}, \varphi, x_3, a_1, a_2, a_3, v_{IN})$ ($Q = IN, INB, M, MB$) as a function of the coordinates x_{12}, x_3 is determined by the following transformations

$$x_n = \frac{x_3}{\cos\theta}, \quad \sin\nu = \frac{x_{12}}{\sqrt{x_{12}^2 + x_3^2}}, \quad \cos\nu = \frac{x_3}{\sqrt{x_{12}^2 + x_3^2}}, \quad \tan\nu = \frac{1}{\cot\nu} = \frac{x_{12}}{x_3}, \tag{9.1}$$

where $\cos\theta$ is given by Equation (1.6).

Cell Matrix. The curve integral W_{cM} of $w_M = w_M(x_{12}, \varphi, x_3, a_1, a_2, a_3, v_{IN})$ along the abscissa P_1P_2 (see Figure 9.1) in the plane $x_{12}x_3$ of the matrix (see Figures 1.4, 1.5) has the form

$$W_{cM} = \int_{P_1P_2} w_M \, dx_3 = \int_0^{d/2} w_M \, dx_3. \tag{9.2}$$

Let $f_{12M} = f_{12M}(x_{12}, \varphi, a_1, a_2, a_3, v_{IN})$ represent a decreasing function of the variable $x_{12} \in \langle a_{12}, x_{0M} \rangle$, which describe a shape of the matrix crack in the plane $x_{12}x_3$ (see Figure 1.4), where $\varphi \in \langle 0, \pi/2 \rangle$, a_1, a_2, a_3, v_{IN} are parameters of this decreasing function. As presented in [3, 4, 5, 19]–[22], we get

$$\frac{\partial f_{12M}}{\partial x_{12}} = -\frac{\sqrt{W_{cM}^2 - \vartheta_M^2}}{\vartheta_M}, \tag{9.3}$$

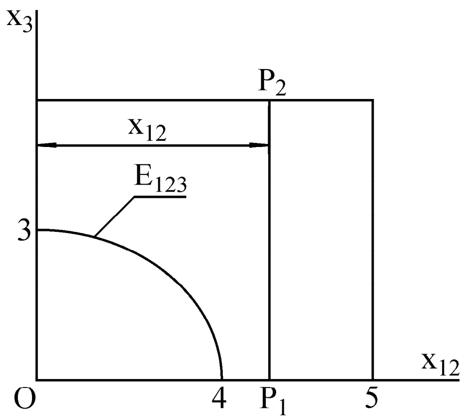


Figure 9.1: The ellipse E_{123} and the abscissa P_1P_2 in the plane $x_{12}x_3$ of the cubic cell (see Figures (1.4), (1.5)), where $a_{12} = O4$, $x_{122} = O5$ are given by Equation (1.9), and $a_3 = O3$.

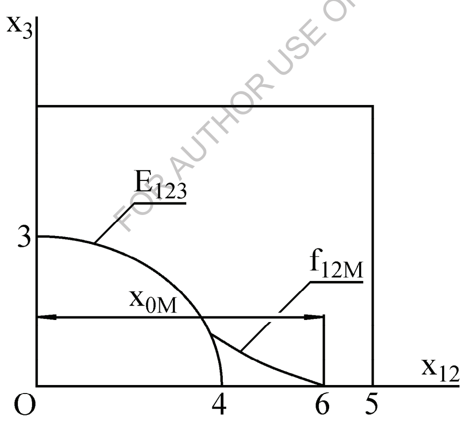


Figure 9.2: The decreasing function $f_{12M} = f_{12M}(x_{12}, \varphi, a_1, a_2, a_3, v_{IN})$ of the variable $x_{12} \in \langle a_{12}, x_{0M} \rangle$, which describes a shape of the matrix crack in the plane $x_{12}x_3$ (see Figure 1.4) for $a_{12} > a_{12M}^{(IC)}$ or $a_{12} > a_{12M}^{(TC)}$ (see Equations (9.8), (9.9)), where $x_{0M} = x_{0M}(\varphi)$ defnes a position of the crack tip in the matrix, and $\varphi \in \langle 0, \pi/2 \rangle$, a_1, a_2, a_3, v_{IN} are parameters of this decreasing function.

where ϑ_M is energy per unit length in the matrix. In case of intercrystalline crack formation, we get

$$\vartheta_M = \frac{K_{ICM}^2}{E_M}, \tag{9.4}$$

where K_{ICM} is fracture toughness of the matrix. In case of transcrystalline crack formation, we get

$$\vartheta_M = \vartheta_{gbM}, \tag{9.5}$$

where the energy ϑ_{gbM} per unit length is related to the inter-atomic bonding of boundaries of crystalline grain in the matrix.

As presented in [3, 4, 5, 19]–[22], the condition

$$(W_{cM})_{x_{12}=a_{12}} - \vartheta_M = 0, \tag{9.6}$$

is a transcendental equation with the variable a_{12} and the parameters $\varphi \in \langle 0, \pi/2 \rangle$, a_1, a_2, a_3, v_{IN} (see Figure 1.4.

The roots $a_{12M}^{(IC)} = a_{12M}^{(IC)}(\varphi, a_1, a_2, a_3, v_{IN})$ and $a_{12M}^{(TC)} = a_{12M}^{(TC)}(\varphi, a_1, a_2, a_3, v_{IN})$ (see Equation (1.7)) of Equation (9.3) for ϑ_M, which is given by Equations (9.4) and (9.5), represents such a dimension of the ellipsoidal inclusion along the axis $x_{12} \subset x_1x_2$ (see Figures 1.4, 1.5), which is critical with respect to the intercrystalline and transcrystalline crack formation in the plane x_1x_2, respectively. Accordingly, if $a_{12M}^{(IC)} > a_{12M}^{(TC)}$ or $a_{12M}^{(IC)} < a_{12M}^{(TC)}$, then the intercrystalline or transcrystalline matrix crack is formed in the plane x_1x_2, respectively.

Let the function $a_{12M}^{(X)} = a_{12M}^{(X)}(\varphi, a_1, a_2, a_3, v_{IN})$ (X=IC,TC) of the variable $\varphi \in \langle 0, \pi/2 \rangle$ exhibit the minimum $a_{minM}^{(X)}$ for $\varphi = \varphi_{minM}^{(X)}$. The critical dimension $a_{minM}^{(X)} = a_{minM}^{(X)}(a_1, a_2, a_3, v_{IN})$ (X=IC,TC) along the axis $x_{12} \subset x_1x_2$ (see Figures 1.4, 1.5) defnes a limit state with respect to the formation of the intercrystalline matrix crack (X=IC) and the transcrystalline matrix crack (X=TC) in the plane x_1x_2 at the microstructural parameters a_1, a_2, a_3, v_{IN} (see Equation (1.1)). Accordingly, if $a_{12} > a_{12M}^{(X)}$ (X=IC,TC), the condition [3, 4, 5, 19]–[22]

$$W_{cM} - \vartheta_M = 0, \quad a_{12} > a_{12M}^{(X)}, \quad X = IC, TC \tag{9.7}$$

represents a transcendental equation with the variable x_{12} and with the root $x_{0M} = x_{0M}(\varphi, a_2, a_3, v_{IN})$, which defnes a position of the crack tip in the matrix (see Figure 9.2). Consequently, the decreasing function $f_{12M} = f_{12M}(x_{12}, \varphi, a_1, a_2, a_3, v_{IN})$ with the variable $x_{12} \in \langle a_{12}, x_{0M} \rangle$ and with the parameters $\varphi \in \langle 0, \pi/2 \rangle$, a_1, a_2, a_3, v_{IN} (see Figures 1.4, 1.5), which describes a shape of the matrix crack in the plane $x_{12}x_3$ for $a_{12} > a_{12M}^{(X)}$ ((X=IC,TC)), has the form [3, 4, 5, 19]–[22]

$$f_{12M} = \frac{1}{\vartheta_M}\left[C_M - \int\left(\sqrt{W_{cM}^2 - \vartheta_M^2}\right) dx_{12}\right], \quad x_{12} \in \langle a_{12}, x_{0M}\rangle, \tag{9.8}$$

where $C_M = C_M(\varphi, a_1, a_2, a_3, v_{IN})$ is derived as [3, 4, 5, 19]–[22]

$$C_M = \left[\int\left(\sqrt{W_{cM}^2 - \vartheta_M^2}\right) dx_{12}\right]_{x_{12}=x_{0M}}. \tag{9.9}$$

Ellipsoidal Inclusion. The curve integral W_{cIN} of $w_Q = w_Q(x_{12}, \varphi, x_3, a_1, a_2, a_3, v_{IN})$ ($Q = IN, INB$) along the abscissa P_1P_2 (see Figure 9.3) in the plane $x_{12}x_3$ of the ellipsoidal inclusion (see Figures 1.4, 1.5) has the form

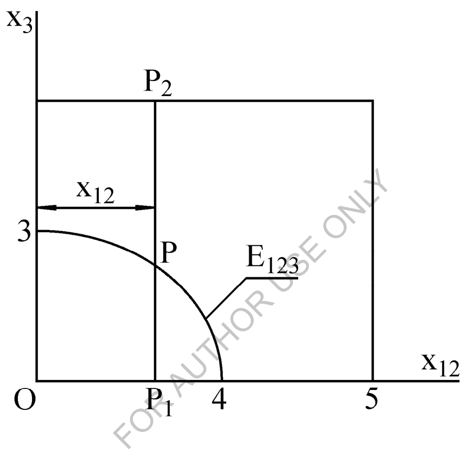


Figure 9.3: The ellipse E_{123} and the abscissa P_1PP_2 in the plane $x_{12}x_3$ of the cubic cell (see Figures (1.4), (1.5)), where $a_{12} = O4$, $x_{122} = O5$ are given by Equation (1.9), and $a_3 = O3$.

$$W_{cIN} = \int_{P_1P} w_{IN}\, dx_3 + \int_{PP_2} w_M\, dx_3 = \int_0^{b_1} w_{IN}\, dx_3 + \int_{b_1}^{d/2} w_M\, dx_3, \tag{9.10}$$

where $a_{12} = O4$ (see Equation (1.7)), $a_3 = O3$, and b_1 is derived as (see Equation (1.2))

$$b_1 = P_1P = \frac{a_3\sqrt{a_{12}^2 - x_{12}^2}}{a_{12}}, \quad x_{12} \in \langle 0, a_{12}\rangle. \tag{9.11}$$

With regard to the intercrystalline and transcrystalline inclusion cracks (see Figure 9.4), the sign '-' and the subscript M in Equations (9.3) and (9.3)–(9.7) are replaced by the sign '+' and the subscript IN, respectively.

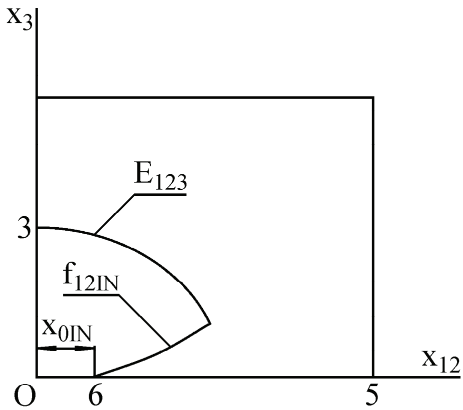


Figure 9.4: The increasing function $f_{12IN} = f_{12IN}(x_{12}, \varphi, a_1, a_2, a_3, v_{IN})$ of the variable $x_{12} \in \langle a_{12}, x_{0IN} \rangle$, which describes a shape of the inclusion crack in the plane $x_{12}x_3$ (see Figure 1.4) for $a_{12} > a_{12IN}^{(IC)}$ or $a_{12} > a_{12IN}^{(TC)}$ (see Equations (9.8), (9.9)), where $x_{0IN} = x_{0IN}(\varphi)$ defnes a position of the crack tip in the inclusion, and $\varphi \in \langle 0, \pi/2 \rangle$ is a parameter of this increasing function.

Consequently, the increasing function $f_{12IN} = f_{12IN}(x_{12}, \varphi, a_1, a_2, a_3, v_{IN})$ with the variable $x_{12} \in \langle a_{12}, x_{0IN} \rangle$ and with the parameters $\varphi \in \langle 0, \pi/2 \rangle$, a_1, a_2, a_3, v_{IN} (see Figures 1.4, 1.5), which describes a shape of the inclusion crack in the plane $x_{12}x_3$ for $a_{12} > a_{12IN}^{(IC)}$ or $a_{12} > a_{12IN}^{(TC)}$, has the form [3, 4, 5, 19]–[22]

$$f_{12IN} = \frac{1}{\vartheta_{IN}} \left[\int \left(\sqrt{W_{cIN}^2 - \vartheta_{IN}^2} \right) dx_{12} - C_{IN} \right], \quad x_{12} \in \langle a_{12}, x_{0IN} \rangle, \tag{9.12}$$

where $C_{IN} = C_{IN}(\varphi, a_1, a_2, a_3, v_{IN})$ is derived as [3, 4, 5, 19]–[22]

$$C_{IN} = \left[\int \left(\sqrt{W_{cIN}^2 - \vartheta_{IN}^2} \right) dx_{12} \right]_{x_{12} = x_{0IN}}. \tag{9.13}$$

Chapter 10

Appendix

Cramer's Rule. The system of n linear algebraic equations is derived as

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ &a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_2, \\ &\;\vdots \qquad\;\; \vdots \qquad\; \vdots \qquad\;\; \vdots \\ &a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n. \end{aligned} \tag{10.1}$$

The root x_i ($i = 1, \ldots$, n) is determined by Cramer's rule [23]

$$x_i = \frac{D_i^{(n)}}{D^{(n)}}, \quad i = 1, \ldots, n, \tag{10.2}$$

where the determinant $D^{(n)}$ with n rows and n columns has the form

$$\begin{aligned} D^{(n)} &= \begin{vmatrix} a_{11}, & a_{12}, & \ldots & a_{1n} \\ a_{21}, & a_{22}, & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}, & a_{n2}, & \ldots & a_{nn} \end{vmatrix} \\ &= \sum_{i=1}^{n} (-1)^{1+i} a_{1i} D_{1i}^{(n-1)} = \sum_{i=1}^{n} (-1)^{1+i} a_{i1} D_{i1}^{(n-1)}. \end{aligned} \tag{10.3}$$

The subdeterminant $D_i^{(n)}$ is created from $D^{(n)}$, i.e., the i-th column of $D^{(n)}$ is replaced by

$$\left.\begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{matrix}\right\} n \text{ rows.} \tag{10.4}$$

Similarly, the subdeterminant $D_{ij}^{(n-1)}$ ($i, j = 1, \ldots$, n) with $(n-1)$ rows and $(n-1)$ columns is created from $D^{(n)}$, i.e., the i-th row and the j-th column of $D^{(n)}$ are omitted. If $n = 2$, then we get

$$D^{(2)} = \begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \tag{10.5}$$

Consequently, if $n = 3$, then we get

$$\begin{aligned} D^{(4)} &= \begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix} \\ &= a_{11} \begin{vmatrix} a_{22}, & a_{23} \\ a_{32}, & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21}, & a_{23} \\ a_{31}, & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21}, & a_{22} \\ a_{31}, & a_{32} \end{vmatrix}. \end{aligned} \tag{10.6}$$

Integrals. The derivatives of the functions $f = x^{\lambda}$, $f = \ln x$ and the constant C are derived as [23]

$$\left(x^{\lambda}\right)' = \lambda x^{\lambda-1}, \quad (\ln x)' = \frac{1}{x}, \quad C' = 0, \tag{10.7}$$

The indefinite integrals of $f = x^{\lambda}$, $f = \ln x$ and the constant C have the forms [23]

$$\int x^{\lambda}\,dx = \frac{x^{\lambda}+1}{\lambda+1}, \quad \lambda \neq -1; \int \frac{dx}{x} = \ln x, \quad \int C\,dx = Cr. \tag{10.8}$$

In case of the product fg of the functions $f = f(x)$, $g = g(x)$, we get [23]

$$(fg)' = f'g + fg'. \tag{10.9}$$

and then the integral of fg has the form [23]

$$\int f'g\,dx = fg - \int fg'\,dx. \tag{10.10}$$

With regard to Equation (10.17), the following integrals are derived as [23]

$$\begin{aligned} &\int x^{\lambda} \ln x\,dx = \frac{x^{\lambda+1}}{\lambda+1} \ln x - \int \frac{x^{\lambda+1}}{\lambda+1} \times \frac{1}{x}\,dx = \frac{x^{\lambda+1}}{\lambda+1} \ln x - \frac{1}{\lambda+1} \int x^{\lambda}\,dx \\ &\quad = \frac{x^{\lambda+1}}{\lambda+1} \left(\ln x - \frac{1}{\lambda+1}\right), \quad \lambda \neq -1, \\ &\int \ln x\,dx = \int 1 \times \ln x\,dx = x \ln x - \int x \times \frac{1}{x}\,dx = x \ln x - \int 1 \times dx = x(\ln x - 1), \\ &\int x^{\lambda} \ln^2 x\,dx = \frac{1}{\lambda+1} \left[x^{\lambda+1} \ln^2 x - 2 \int x^{\lambda} \ln x\,dx\right] \end{aligned}$$

$$= \frac{x^{\lambda+1}}{\lambda+1}\left[\left(\ln x - \frac{1}{\lambda+1}\right)^2 + \frac{1}{(\lambda+1)^2}\right], \quad \lambda \neq -1. \tag{10.11}$$

Let $F = F(x)$ be a primitive function of $f = f(x)$ in the interval $x \in \langle a,b \rangle$, i.e., $f = dF/dx$. The definite integral $\int_a^b f\,dx$ is defined by Newton-Leibniz's formula [23], which has the form

$$\int_a^b f\,dx = F(b) - F(a). \tag{10.12}$$

Wronskian's Method. The differential equation (4.3) with a non-zero right-hand side [23] is derived as

$$\frac{\partial^2 u_n}{\partial x^2} + \frac{2}{x_n}\frac{\partial u_n}{\partial x_n} - \frac{2u_n}{x_n^2} = g, \quad g = \sum_{i=1}^{3} C_i x^{\kappa_i - 2}, \tag{10.13}$$

where the integration constants C_1, C_2, C_3 are determined by the mathematical boundary conditions in Section 2.3. If $g = 0$, we get

$$\frac{\partial^2 u_n}{\partial x^2} + \frac{2}{x_n}\frac{\partial u_n}{\partial x_n} - \frac{2u_n}{x_n^2} = 0. \tag{10.14}$$

If $u_n = x^\lambda$, then the solutions u_{1n}, u_{2n} of Equation (10.24) have the forms

$$u_{1n} = x_n, \quad u_{2n} = \frac{1}{x_n^2}. \tag{10.15}$$

The solution u_n of Equation (10.22) is derived as [23]

$$u_n = \sum_{i=1}^{2} a_i u_{in}, \quad a_i = \int \frac{W_i^{(2)}}{W^{(2)}}\,dx_n, \quad i = 1,2. \tag{10.16}$$

Wronskian's determinants $W^{(2)}$, $W_i^{(2)}$ $(i = 1,2)$ with 2 rows and 2 columns are have the forms [23]

$$W^{(2)} = \begin{vmatrix} u_{1n}, & u_{2n} \\ \frac{\partial u_{1n}}{\partial x_n}, & \frac{\partial u_{2n}}{\partial x_n} \end{vmatrix}, \quad W_1^{(2)} = \begin{vmatrix} 0, & u_{2n} \\ \mathrm{g}, & \frac{\partial u_{2n}}{\partial x_n} \end{vmatrix}, \quad W_2^{(2)} = \begin{vmatrix} u_{1n}, & 0 \\ \frac{\partial u_{1n}}{\partial x_n}, & \mathrm{g} \end{vmatrix}. \tag{10.17}$$

The determinant $W_i^{(2)}$ $(i = 1,2)$ is created from $W^{(2)}$, i.e., the i-th column of $W^{(2)}$ is replaced by the following one [23]

$$\left.\begin{matrix} 0 \\ g \end{matrix}\right\} \text{2 rows.} \tag{10.18}$$

Let $f_1, \ldots, f_n$ represent n solutions of a differential equation of the n-th rank with zero right-hand side. Let the functions $f_1, \ldots, f_n$ of the variable x exhibit continuous derivatives to the $(n-1)$-th degree. The solution of this differential equation with a non-zero right-hand side (i.e., $g \neq 0$) is derived as [23]

$$f = \sum_{i=1}^{n} a_i f_i, \quad a_i = \int \frac{W_i^{(n)}}{W^{(n)}}\, dx. \tag{10.19}$$

With respect to $f_1, \ldots, f_n$, Wronskian's determinant $W^{(n)}$ ($i = 1, \ldots, \mathrm{n}$) with n rows and n columns have the form [23]

$$W^{(n)} = \begin{vmatrix} f_1, & f_2, & \cdots & f_n \\ \dfrac{\partial f_1}{\partial x}, & \dfrac{\partial f_2}{\partial x}, & \cdots & \dfrac{\partial f_n}{\partial x} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial^{n-1} f_1}{\partial x^{n-1}}, & \dfrac{\partial^{n-1} f_2}{\partial x^{n-1}}, & \cdots & \dfrac{\partial^{n-1} f_n}{\partial x^{n-1}} \end{vmatrix}, \tag{10.20}$$

where $W_i^{(n)}$ ($i = 1, \ldots, \mathrm{n}$) with n rows and n columns is created from $W^{(n)}$, i.e., the i-th column of $W^{(n)}$ is replaced by the following one [23]

$$\left.\begin{matrix} 0 \\ 0 \\ \vdots \\ g \end{matrix}\right\} n \text{ rows.} \tag{10.21}$$

Numerical Determination. Numerical values of the thermal stresses in a real matrix-inclusion composite include integrals and derivatives, which are determined by a programming language. If $f = f(x)$, then a numerical value of the derivative $\partial f / \partial x$ is determined by [23]

$$\frac{\partial f}{\partial x} \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{10.22}$$

In case of the angles φ, ν (see Figure 1.4), the step $\Delta x = \Delta\varphi = \Delta\nu = 10^{-6}$ [deg] is sufficient [3, 4, 5, 19]–[22].

Let F represent a definite integral of the function $f = f(\varphi, \nu)$ with the variables $\varphi, \nu \in \langle 0, \pi/2 \rangle$. Let n, m be integral parts of the real numbers $\pi/(2\Delta\varphi)$, $\pi/(2\Delta\nu)$

[3, 4, 5, 19]–[22], respectively. Numerical values of the definite integral F are determined by the following formula [23], [3, 4, 5, 19]–[22]

$$F = \int_{0}^{\pi/2} \int_{0}^{\pi/2} f(\varphi, \nu)\, d\varphi\, d\nu \approx \sum_{j=0}^{m} \left(\sum_{i=0}^{n} f(i \times \Delta\varphi; j \times \Delta\nu)\, \Delta\varphi \right) \Delta\nu, \tag{10.23}$$

where the steps $\Delta\varphi = \Delta\nu = 0.1$ [deg] are sufficient. Finally, the average numerical value $\overline{f}$ of the function $f = f(\varphi, \nu)$ with the variables $\varphi, \nu \in \langle 0, \pi/2 \rangle$ is determined by the following formula [23]

$$\overline{f} = \left(\frac{2}{\pi}\right)^2 \int_{0}^{\pi/2} \int_{0}^{\pi/2} f(\varphi, \nu)\, d\varphi d\nu \approx \left(\frac{2}{\pi}\right)^2 \sum_{j=0}^{m} \left(\sum_{i=0}^{n} f(i \times \Delta\varphi; j \times \Delta\nu)\, \Delta\varphi \right) \Delta\nu. \tag{10.24}$$

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