Mathematical Models of Stresses in Materials

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This monograph is dedicated with love to
my dearest parents and grandparents.

CONTENTS

INTRODUCTION

This monograph[1] presents original mathematical models of

- thermal and phase-transformation stresses, which originate in matrix-inclusion composites during a cooling process,
- material micro- strengthening and macro-strengthening, which is induced by these stresses,
- intercrystalline and transcrystalline crack formation, including mathematical definitions of critical limit states with respect to the material crack formation, which is induced by these stresses.

The material strengthening and the limit states represent important phenomena in material science and engineering.

The stresses are determined for a multi-inclusion-matrix model system with isotropic ellipsoidal inclusions with the inter-inclusion distance *d*, which are periodically distributed in an isotropic matrix. This model system corresponds to real two-component materials, which consist of

- isotropic ellipsoidal precipitates, distributed in isotropic crystal-line grains (e.g., matrix-precipitate composites),
- two types of isotropic crystalline grains with different material properties (e.g., dual-phase steel).

The thermal stresses are a consequence of different thermal expansion coefficients of the matrix and ellipsoidal inclusions. The phase-transfor-

[1] This monograph was supported by the Slovak scientific grant agency VEGA 2/0069/24.

mation stresses are a consequence of a different dimension of a cubic crystalline lattice, which is transformed in the inclusion and/or matrix.

Mathematical and computational models of phenomena in infinite periodic matrix-inclusion model systems are determined within identical suitable cells, and each cell contains a central component (e.g., an inclusion, a crystalline grain, a pore). Due to this infinity and periodicity, the models, which are determined for a certain cell, are valid for any cell. Infinite matrixes are used due to simplicity of mathematical solutions for material components (e.g., precipitates, pores). The material components are small in comparison with macroscopic material samples or macroscopic structu-ral elements, and then the solutions are acceptable in spite of this simplifi-cation (Mura, 1987, 31-32).

The mathematical models results from fundamental equations of mechanics of a solid continuum, with respect to its shape, loading, mechanical constraints and the principle of minimum potential energy.

The infinite multi-inclusion-matrix model system is imaginarily divi-ded into cubic cells with the dimension *d* and with a central ellipsoidal inclusion, and the stresses are determined within the cubic cell. Mathematical solutions for this multi-inclusion-matrix model system correspond to real composites, in contrast to

- the simple one-inclusion mathematical model in (Selsing, 1961, 419-419), determined for a simple one-inclusion-matrix model system,
- the simple multi-inclusion mathematical model in (Mizutani, 1996, 483-494), determined for physically unacceptable mechani-cal constraints due to unsuitable cells of a multi-inclusion-matrix system.

Different mathematical procedures, which are applied to the fundame-ntal equations (Cauchy's and equilibrium equations, Hooke's law), result in different mathematical solutions for the stresses in the matrix and ellipsoidal inclusion. Finally, such a combination of the different mathematical solutions for the matrix and the ellipsoidal inclusion is considered to exhibit minimum potential energy.

The mathematical models are determined by standard procedures of mechanics of a solid continuum, which include definitions of

- such a multi-inclusion-matrix model system and a coordinate sys-tem, which correspond to real matrix-inclusion composite ma-terials,

- reasons of the thermal and phase-transformation stresses,
- the fundamental equations, which result in a system of differen-tial equations,
- elastic energy density and elastic energy of the model system,
- mechanical constraints, i.e., mathematical boundary conditions, for the matrix and ellipsoidal inclusion,
- different mathematical procedures, which are applied to the sys-tem of the differential equations,
- final formulae for the thermal and phase-transformation stresses, strains, elastic energy density and elastic energy,
- final formulae for the material micro-/macro-strengthening in the matrix and ellipsoidal inclusion,
- mathematical procedures to determine such critical dimensions of the ellipsoidal inclusion, which are reason of a crack in the ma-trix,
- mathematical procedures to determine dimensions of the matrix crack.

In contrast to author's mathematical models (Ceniga, 2008, 10-11; 2007, 9-12) for composites with inclusions of an ideal spherical shape, the mathematical models in this monograph, which are determined for composites with ellipsoidal inclusions, represent a more realistic description of the stress-strain state in real matrix-inclusion composite materials.

The mathematical results in this monograph are then applicable within

- basic research (mechanics of a solid continuum, theoretical physics, material science),
- the engineering practice, i.e., material technology,
- as well as within university undergraduate and postgraduate cour-ses, as a textbook on analytical material mechanics.

With regard to the basic research, the results of this monograph can be incor-porated to mathematical models, which defines the disturbance of an ap--plied stress field around inclusions in a solid continuum (Eshelby, 1957, 376-396), as well as into mathematical, computational and experimental models of overall materials stresses, overall material strengthening, inter-actions of energy barriers with dislocations and domain walls, etc.

The mathematical models include microstructural parameters of a real matrix-inclusion composite (the inclusion dimensions a_{1IN}, a_{2IN}, a_{3IN}, the inclusion volume fraction v_{IN}, the inter-inclusion distance d), and are applicable to composites with ellipsoidal inclusions of different morphology, i.e., $a_{1IN} \approx a_{2IN} \approx a_{3IN}$ (dual-phase steel), $a_{1IN} >> a_{2IN} \approx a_{3IN}$ (martensitic steel).

In case of real two-component materials (the engineering practice), material scientists and engineers can determine such numerical values of the microstructural parameters,

- which result in maximum values of the material micro-and ma-cro-strengthening,
- which define the limit states (i.e., critical states) with respect to the intercrystalline or transcrystalline crack formation in the ma-trix and the ellipsoidal inclusion.

Consequently, the material scientists and engineers can develop sui-table technological processes, which result in such microstructural para-meters to obtain maximum strengthening, and to avoid the crack forma-tion.

This numerical determination, performed by suitable programming languages, result from the mathematical procedure in Appendix.

With respect to the university courses, the fundamental equations of mechanics of a solid continuum, along with the mathematical procedures, are explained and determined in detail. As a textbook on analytical material mechanics, this monograph is then suitable for non-specialists in mechanics of a solid continuum. Finally, Appendix presents such mathematical topics, which are required to perform the mathematical procedures in this monograph.

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CHAPTER 1

MODEL MATERIAL SYSTEM

1.1 Matrix-Inclusion System

Figure 1.1 shows the model material system, which consists of an infi-nite matrix and periodically distributed ellipsoidal inclusions with the di-mensions a_{1IN}, a_{2IN}, a_{3IN} along the axes x_1, x_2, x_3 of the Cartesian system ($Ox_1x_2x_3$), respectively, and with the inter-inclusion distance d along x_1, x_2, x_3. The point O is a centre of the ellipsoidal inclusion.

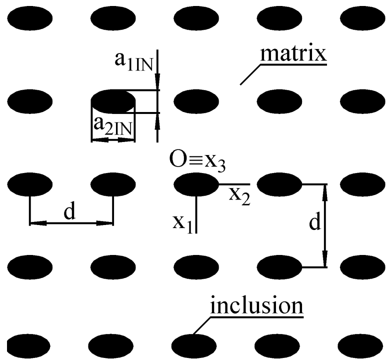


Figure 1.1. The matrix-inclusion system with infinite matrix and periodi-cally distributed ellipsoidal inclusions: the dimensions a_{1IN}, a_{2IN}, a_{3IN} along the axes x_1, x_2, x_3 of the Cartesian system ($Ox_1x_2x_3$), respectively; the inter-inclusion distance d along the axes x_1, x_2, x_3; the inclusion centre O.

The mathematical models of the thermal and phase-transformation stresses are determined in the cubic cell with the dimension d and with a central ellipsoidal inclusion (see Figure 1.2). With regard to the volume $V_{IN} = 4\,\pi\, a_{1IN}\, a_{2IN}\, a_{3IN}$ and $V_C = d^3$ of the ellipsoidal inclusion and the cubic cell, the inclusion volume fraction v_{IN} and the inter-inclusion distance d have the forms

$$v_{IN} = \frac{V_{IN}}{V_C} = \frac{4\pi\, a_{1IN}\, a_{2IN}\, a_{3IN}}{3d^3} \in \left(0, \frac{\pi}{6}\right),$$

$$d = \left(\frac{4\pi\, a_{1IN}\, a_{2IN}\, a_{3IN}}{3v_{IN}}\right)^{1/3}, \tag{1.1}$$

where $v_{INmax} = \pi / 6$ is given by the condition $a_i \rightarrow d/2$ (i = 1,2,3). The inter-inclusion distance $d = d(a_{1IN}, a_{2IN}, a_{3IN}, v_{IN})$ is included within the mechanical constraints (see Equations (1.15), (4.1)-(4.5)), and the stresses are a function of the microstructural parameters a_{1IN}, a_{2IN}, a_{3IN}, v_{IN}, d.

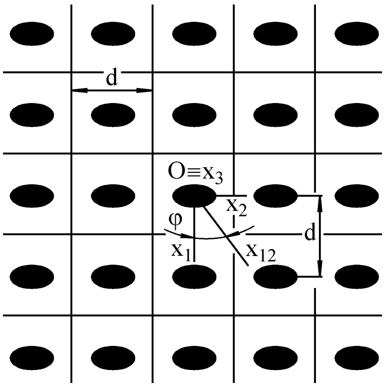


Figure 1.2. The cubic cells with the dimension d and with the plane x_1x_2, where O is a centre of the ellipsoidal inclusions, and $x_{12} \subset x_1x_2$, $x_{12}x_3 \subset x_1x_2$.

This model system corresponds to real two-component materials, which consist of

- isotropic ellipsoidal precipitates, distributed in isotropic crystal-line grains, e.g., matrix-precipitate composites,
- two types of isotropic crystalline grains with different material properties, e.g., dual-phase steel with the grains A and B.

Consequently, the ellipsoidal precipitates and the crystalline grains are considered to represent the ellipsoidal inclusion and the matrix of the model matrix-inclusion system.

Similarly, let the crystal grains *A* and *B* be characterized by the volume fraction v_A and v_B, respectively, where $v_A+v_B=1$. If $v_A<v_B$, then the grains *A* and *B* are considered to represent the ellipsoidal inclusion and the matrix, respectively. If $v_A>v_B$, then the grains *A* and *B* are considered to represent the matrix and the ellipsoidal inclusion, respectively. If $v_A=v_B$, then the following energy analysis is required to be consi-dered.

Let the grains *A* and *B* be considered to represent the ellipsoidal inclu-sion and the matrix with the elastic energy W_{INA} and W_{MB}, which is accu-mulated in the ellipsoidal inclusion and the cell matrix (see Equation (2.30)), respectively.

Let the grains *A* and *B* be considered to represent the matrix and the ellipsoidal inclusion with the elastic energy W_{MA} and W_{INB}, which is accumulated in the cell matrix and the ellipsoidal inclusion (see Equation (2.30)), respectively.

If $W_{INA} + W_{MB} < W_{MA} + W_{INB}$, then the grains *A* and *B* are considered to represent the ellipsoidal inclusion and the matrix, respectively. If $W_{INA} + W_{MB} > W_{MA} + W_{INB}$, the grains *A* and *B* are considered to represent the matrix and the ellipsoidal inclusion, respectively.

Mathematical and computational models of phenomena in infinite pe-riodic matrix-inclusion model systems are determined within identical suitable cells. Due to this infinity and periodicity, the mathematical models of the thermal and phase-transformation stresses in the multi-inclusion model system in Figures 1.1, 1.2, which are determined for a certain cell, are valid for any cell. In general, infinite matrixes are used due to simpli-city of mathematical solutions for material components (e.g., precipitates, crystalline grains, pores). Such mathematical solutions are assumed to ex-hibit sufficient accurancy with respect to material components (e.g., preci-pitates, crystalline grains, pores), which are small in comparison with ma-croscopic material samples and macroscopic structural elements. Finally, the mathematical solutions are acceptable in spite of this simplification (Mura, 1987, 31-32).

1.2 Coordinate System

The ellipse *E* with the dimensions *a*, *b* along the axes *x*, *y* of the Carte-sian system (*Oxy*), respectively, is described by the function (Rektorys, 1973, 147)

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1,\quad x=a\cos\alpha,\quad x=b\sin\alpha\,, \tag{1.2}$$

where x, y are coordinates of any point P of the ellipse E. The normal n at the point P has the form (Rektorys, 1973, 148)

$$\frac{\partial x}{\partial\alpha}\left(x-a\cos\alpha\right)+\frac{\partial y}{\partial\alpha}\left(x-b\sin\alpha\right)=0\,. \tag{1.3}$$

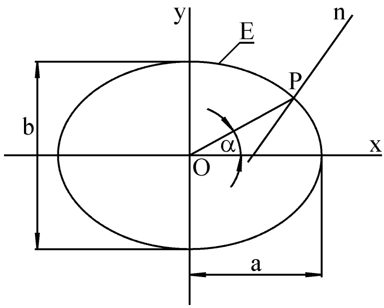


Figure 1.3. The ellipse E with the dimensions a, b along the axes x, y of the Cartesian system (Oxy), respectively, and the P related to the angle α.

With regard to Equations (1.2), (1.3), we get

$$y=\frac{1}{b}\left[x\,a\tan\alpha-\left(a^2-b^2\right)\sin\alpha\right]. \tag{1.4}$$

The stresses are determined by the spherical coordinates (x_n,φ,ν), where $x_n=|P_{12}P|$, $P_{12}\subset x_{12}$ (see Figure 1.4). Equations (1.12), (1.13) define a fuction $\theta=f(\nu)$ for the angles $\theta,\nu\in\langle 0,\pi/2\rangle$. The model system is symmetric (see Figure 1.1, 1.2), and the stresses are sufficient to be determined within the interval $\varphi,\nu\in\langle 0,\pi/2\rangle$. Figure 1.4 shows the ellipsoidal inclusion for φ, $\nu\in\langle 0,\pi/2\rangle$, where $a_{1IN}=O1$, $a_{2IN}=O2$, $a_{3IN}=O3$. With regard to Equation (1.2), any point of the ellipse E_{12} in the plane x_1x_2 has the coordinates

$$x_1=a_{1IN}\cos\varphi,\quad x_2=a_{2IN}\sin\varphi,\quad \varphi\in\left\langle 0,\frac{\pi}{2}\right\rangle. \tag{1.5}$$

The point P of the ellipse E_{123} in $x_{12}x_3$ is described by the coordinates

$$x_{12P} = a_{12} \sin\nu, \quad x_{3P} = a_3 \cos\nu\,,$$
$$a_{12} = O4 = \sqrt{a_{1IN}^2 \cos^2\varphi + a_{2IN}^2 \sin^2\varphi}, \quad \varphi,\nu \in \left\langle 0, \frac{\pi}{2} \right\rangle. \tag{1.6}$$

where ($Px_nx_\varphi x_\theta$) is a Cartesian system at the point P; the axes x_n and x_θ represents a normal and a tangent of the ellipse E_{123} at the point P, respectively; and $x_{12}x_3 \perp x_1x_2$, $x_{12} \subset x_1x_2$, $x_\varphi \perp x_{12}$.

Figure 1.5 shows the cross section $O567$ of the cubic cell in the plane $x_{12}x_3$ (see Figures 1.2, 1.4). The angle $\nu\in\langle 0, \pi/2\rangle$ defines a position of the point P with the Cartesian system ($Px_nx_\varphi x_\theta$) (see Figure 1.4) for $\nu = \nu_0$ (see Figure 1.5a), $\nu\in(0, \nu_0\rangle$ (see Figure 1.5b), $\nu\in\langle \nu_0, \pi/2\rangle$ (see Figure 1.5c). The points P_1, P_2 represent intersections of the normal x_n with O567.

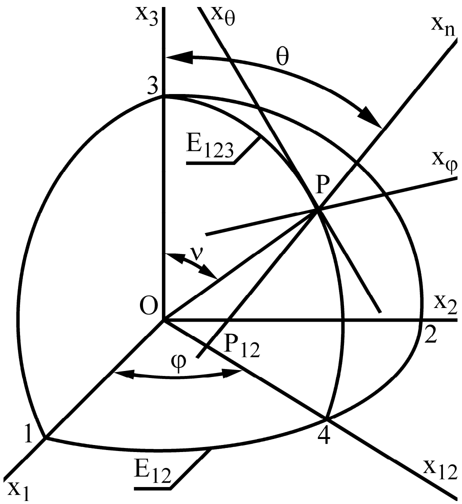


Figure 1.4. The inclusion with the centre O and with the dimensions $a_{1IN} = O1$, $a_{2IN} = O2$, $a_{3IN} = O1$ along the axes x_1, x_2, x_3, respectively. The ellipses E_{12}, E_{123} in the planes x_1x_2, $x_{12}x_3$ (see Figure 1.2) are given by Equations (1.5), (1.6), respectively, where $x_{12}x_3 \perp x_1x_2$, $x_{12} \subset x_1x_2$, $x_\varphi \perp x_{12}$. The point P on the inclusion surface is defined by φ, $\nu\in\langle 0, \pi/2\rangle$, and ($Px_nx_\varphi x_\theta$) is a Cartesian system at the point P, where $P \subset E_{123}$. The axes x_n and x_θ represents a normal and a tangent of the ellipse E_{123} at the point P, where $x_n=|P_{12}P|$, $P_{12}\subset x_{12}$. The function $\theta = f(\nu)$ is given by Equations (1.12), (1.13).

With regard to Equation (1.4), the normal x_n at the point P of the ellipse E_{123} in the plane $x_{12}x_3$ is derived as

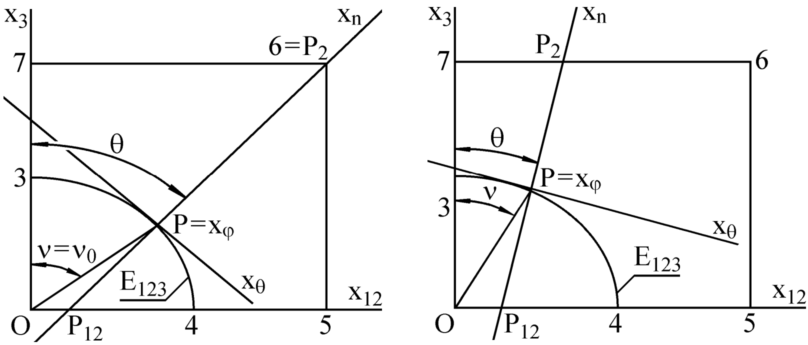
$$x_3 = \frac{\cos\nu}{a_3}\left(\frac{a_{12}\,x_{12}}{\sin\nu} + a_3^2 - a_{12}^2\right), \quad \nu \in \left\langle 0, \frac{\pi}{2}\right\rangle. \tag{1.7}$$

With regard to Equation (1.7), the coordinates x_{121}, x_{31} of the point P_1 is derived as

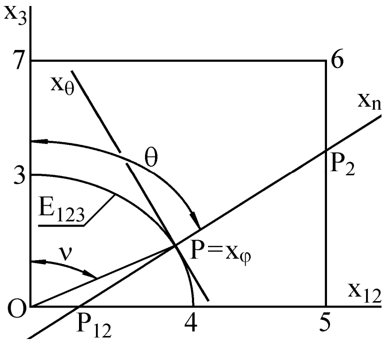
$$x_{121} = \frac{\left(a_{12}^2 - a_3^2\right)\sin\nu}{a_{12}}, \quad x_{31} = 0, \quad \nu \in \left\langle 0, \frac{\pi}{2}\right\rangle, \tag{1.8}$$

The coordinates x_{122}, x_{32} of the point P_2 in Figure 1.5b for $\nu \in (0, \nu_0\rangle$ are derived as

$$x_{122} = \frac{\sin\nu}{a_{12}}\left(\frac{d\cos\nu}{2a_3} + a_{12}^2 - a_3^2\right), \quad x_{32} = \frac{d}{2} \quad \nu \in \left\langle 0, \nu_0\right\rangle. \tag{1.9}$$



(a) (b)



(c)

Figure 1.5. The angle $\nu\in\langle 0,\pi/2\rangle$ defines a position of the point P with the Cartesian system ($Px_n x_\varphi x_\theta$) (see Figure 1.4) for (a) $\nu\in\langle 0,\pi/2\rangle$, (b) $\nu\in(0,\nu_0\rangle$ (c) $\nu\in\langle\nu_0,\pi/2\rangle$, where ν_0 is given by Equation (1.7). The points P_{12} (see Figure 1.4), P_2 represent intersections of the normal x_n with $O567$, where $O567$ is a cross section of the cubic cell in the plane $x_{12}x_3$ (see Figures 1.2, 1.4). The angle $\theta=\angle(x_n,x_3)$ is given by Equation (1.11).

The coordinates x_{122}, x_{32} of the point P_2 in Figure 1.5c for $\nu\in\langle\nu_0,\pi/2\rangle$ have the forms

$$x_{122}=\frac{d}{2c_\varphi\sin\nu},$$

$$c_\varphi=\cos\varphi,\quad \varphi\in\left\langle 0,\frac{\pi}{4}\right\rangle;\quad c_\varphi=\sin\varphi,\quad \varphi\in\left\langle\frac{\pi}{4},\frac{\pi}{2}\right\rangle$$

$$x_{32}=\frac{\cos\nu}{a_3}\left[\frac{a_{12}\,d}{2f(\varphi)\sin\nu}+a_3^2-a_{12}^2\right],\quad \nu\in\left\langle\nu_0,\frac{\pi}{2}\right\rangle. \tag{1.10}$$

The coordinate x_{122} of the point P_2 in Figure 1.5a for $\nu\in(0,\nu_0\rangle$ is given by Equation (1.10), where $x_{32}=d/2$. With regard to Equation (1.7), the angle ν_0 represents a root of the following equation

$$\frac{\cos\nu_0}{a_3}\left[\frac{a_{12}\,d}{2f(\varphi)\sin\nu_0}+a_3^2-a_{12}^2\right]-\frac{d}{2}=0,$$

$$f(\varphi)=\cos\varphi,\quad \varphi\in\left\langle 0,\frac{\pi}{4}\right\rangle;\quad f(\varphi)=\sin\varphi,\quad \varphi\in\left\langle\frac{\pi}{4},\frac{\pi}{2}\right\rangle, \tag{1.11}$$

and this root is determined by a numerical method. The angle $\theta=\angle(x_n,x_3)$ has the form

$$\cos\theta = \frac{x_{3P}}{\sqrt{\left(x_{12P} - x_{121}\right)^2 + x_{3P}^2}} = \frac{1}{\sqrt{1 + \left(\dfrac{a_3 \tan\nu}{a_{12}}\right)^2}} ,$$

$$\sin\theta = \frac{1}{\sqrt{1 + \left(\dfrac{a_3 \cot\nu}{a_{12}}\right)^2}} . \tag{1.12}$$

and then we get

$$\sin\theta\, d\theta = \Omega\, d\nu, \quad \Omega = \frac{1}{\sqrt{\left(\dfrac{a_3}{a_{12}}\right)^2 + \cot^2\nu}\;\left[\left(\dfrac{a_3}{a_{12}}\right)^2 + \cot^2\nu\right]\sin^2\nu} ,$$

$$\frac{\partial}{\partial\theta} = \left(\frac{\partial\theta}{\partial\nu}\right)^{-1}\frac{\partial}{\partial\nu} = \Theta\frac{\partial}{\partial\nu}, \quad \Theta = \frac{a_{12}}{a_3}\left[\left(\frac{a_3 \sin\nu}{a_{12}}\right)^2 + \cos^2\nu\right]. \tag{1.13}$$

The model system in Figure 1.1 is symmetric. Due to this symmetry, any point P on the matrix-inclusion boundary exhibit the normal displace-ment u_{n} along the axis x_{n}. Consequently, any point P of the normal x_{n} exhi-bits u_{n}, and then we get $u_{\varphi} = u_{\theta}$, where u_{φ}, u_{θ} are displacement along the axes x_{φ}, x_{θ}, respectively.

The stresses are determined along the axes x_{n}, x_{φ}, x_{θ} of the Cartesian system ($P\ x_{\mathrm{n}}\ x_{\varphi}\ x_{\theta}$), and represent functions of the spherical coordinates (x_{n}, φ, θ) for $\varphi, \theta \in \langle 0, \pi/2\rangle$. The intervals $x_{\mathrm{n}} \in \langle 0, x_{\mathrm{IN}}\rangle$ and $x_{\mathrm{n}} \in \langle x_{\mathrm{IN}}, x_{\mathrm{M}}\rangle$ are related to the ellipsoidal inclusion and the cell matrix, where $P = P_1$, $P \subset E_{123}$ and $P = P_2$ for $x_{\mathrm{n}} = 0$, $x_{\mathrm{n}} = x_{\mathrm{IN}}$ and $x_{\mathrm{n}} = x_{\mathrm{M}}$ (see Figure 1.5), respectively. Finally, we get

$$x_{IN} = P_1P = \sqrt{\left(x_{12P} - x_{121}\right)^2 + x_{3P}^2} = a_3\sqrt{\left(\frac{a_3 \sin\nu}{a_{12}}\right)^2 + \cos^2\nu} ,$$

$$x_M = PP_2 = \sqrt{\left(x_{122} - x_{12P}\right)^2 + \left(x_{32} - x_{3P}\right)^2}$$

$$= \sqrt{\left(\frac{\sin\nu}{a_{12}}\right)^2 \left(\frac{d\cos\nu}{2a_3} - a_3^2\right)^2 + \left(\frac{a_{12}\cos\nu}{a_3}\right)^2 \left[\frac{d}{2f(\varphi)\sin\nu} - a_{12}\right]^2} \,. \tag{1.15}$$

CHAPTER 2

FUNDAMENTAL EQUATIONS

Fundamental equations of mechanics of a solid continuum are repre-sented by Cauchy's and equilibrium equations, along with Hooke's law (see Section 2.1-2.3), which result in a system of differential equations (see Section 2.4). Due to different mathematical solutions of this systems, which are determined by different mathematical procedures (see Sections 5.1,6.1,7.1,8.1), the analysis of elastic energy density is considered (see Section 2.8).

2.1 Cauchy's Equations

Cauchy's equations define relationships between strains and displacements, and are determined for a suitable infinitesimal part of the model system with respect to a coordinate system (see Figures 1.1, 1.2).

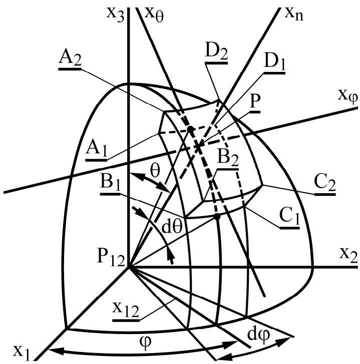


Figure 2.1. The infinitesimal spherical cap at the point P with the surface $S_1 = A_1B_1C_1D_1$ and $S_2 = A_2B_2C_2D_2$ for $x_n = P_{12}P$ (see Figure 1.4) and $x_n{+}dx_n$, respectively, where $A_1A_2{=}B_1B_2{=}C_1C_2{=}D_1D_2{=}dx_n$, $A_1D_1{=}B_1C_1{=}\ x_n{\times}d\varphi$, $A_1B_1{=}C_1D_1{=}x_n{\times}d\theta$, $A_2D_2{=}B_2C_2{=}(x_n{+}dx_n){\times}d\varphi$, $A_2B_2{=}C_2D_2{=}(x_n{+}dx_n)\ {\times}d\theta$. The axes x_n and x_φ, x_θ represent normal and tangential directions (see Figure 1.4), respetively.

Due to the spherical coordinates (r,φ,ν) (see Figure 1.4), the infinitesimal part at the point P is represented by the infinitesimal spheri-cal cap with the dimension $A_1A_2=B_1B_2=C_1C_2=D_1D_2=dx_n$ along the axis x_n, and with the dimensions $A_1A_2=B_1B_2=C_1C_2=D_1D_2=dx_n$, $A_1D_1=B_1C_1=x_n\times d\varphi$, $A_1B_1=C_1D_1=x_n\times d\theta$ and $A_2D_2=B_2C_2=(x_n+dx_n)\times d\varphi$, $A_2B_2= C_2D_2=(x_n+dx_n)\times d\theta$ along the axes x_φ, x_θ for $x_n = P_{12}P$ (see Figure 1.4) and x_n+dx_n, respecti-vely. The axes x_n and x_φ, x_θ represent normal and tangential directions (see Figure 1.4), respetively.

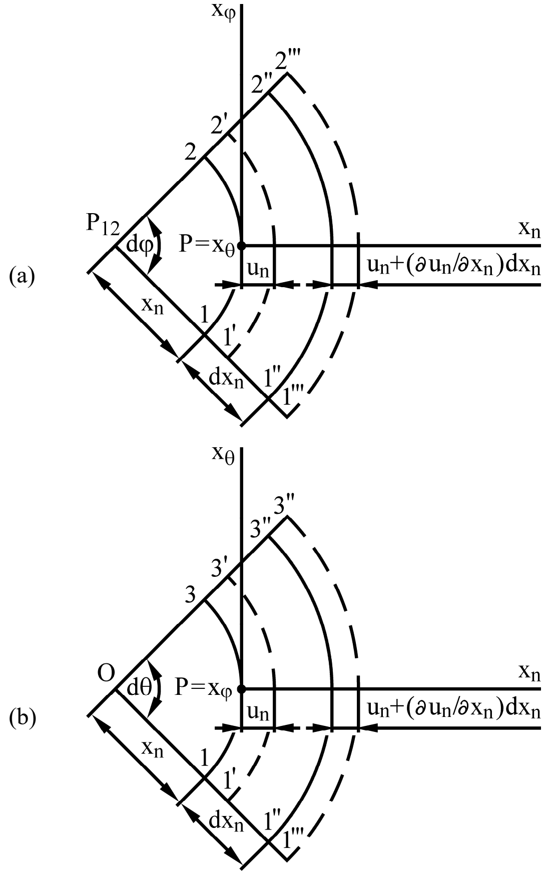


Figure 2.2. The normal displacement u_n and $u_n+(\partial u_n/\partial x_n)\ dx_n$ of the infini-tesimal spherical cap at the point P for $x_n = P_{12}P$ and x_n+dx_n in the plane (a) x_nx_φ, (b) $x_n\ x_\theta$ (see Figures 1.4, 2.1), respectively.

As analysed in Chapter 1, any point P of the normal x_n (see Figure 1.4) exhibits the normal displacement u_n along the normal x_n, and then we get $u_\varphi = u_\theta$, where u_φ, u_θ are displacement along the axes x_φ, x_θ, respectively. The stresses are determined along the axes x_n, x_φ, x_θ of the Cartesian system ($Px_nx_\varphi x_\theta$).

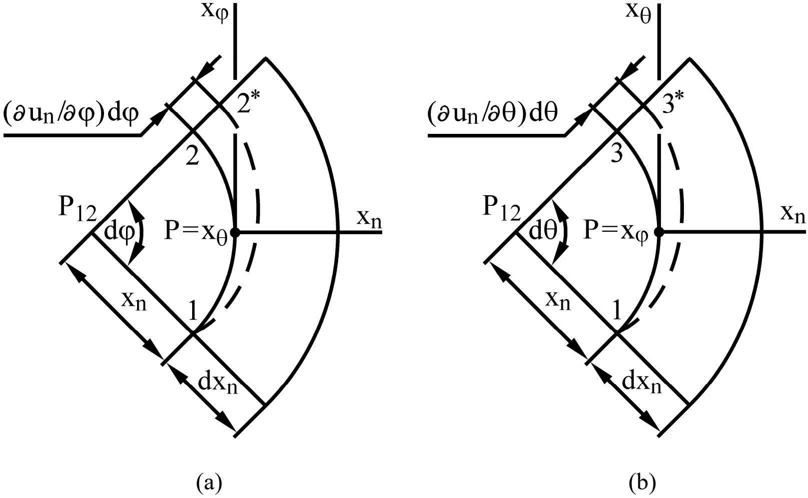


Figure 2.3. The normal displacement $u_n = u_n(\varphi,\theta)$ of the infinitesimal sphe-rical cap at the point P in the plane (a) $x_n x_\varphi$, (b) $x_n\, x_\theta$ (see Figures 1.4, 2.1).

With regard to Figure 2.2, the normal strain ε_n along the axis x_n, and the tangential strains ε_φ, ε_θ along the axes x_φ, x_θ, respectively, are derived as (Ceniga, 2007, 35-38; Ceniga, 2008, 23-26)

$$\varepsilon_n = \frac{\left|1'''1'\right| - \left|1''1'\right|}{\left|1''1'\right|} = \frac{1}{dx_n}\left[\left(dx_n + \frac{\partial u_n}{\partial x_n}\,dx_n\right) - dx_n\right] = \frac{\partial u_n}{\partial x_n}, \qquad (2.1)$$

$$\varepsilon_\varphi = \varepsilon_\theta = \frac{\left|1'2'\right| - \left|12\right|}{\left|12\right|} = \frac{\left|1'3'\right| - \left|13\right|}{\left|13\right|}$$
$$= \frac{(u_n + x_n)\,d\varphi - x_n\,d\varphi}{x_n\,d\varphi} = \frac{(u_n + x_n)\,d\theta - x_n\,d\theta}{x_n\,d\theta} = \frac{u_n}{x_n}, \qquad (2.2)$$

where $|13| = x_n\, d\theta$, $|1'3'| = (u_n + x_n)\, d\theta$ is considered instead of $|13| = x_n \sin\theta\, d\theta$, $|1'3'| = (u_n + x_n)\sin\theta\, d\theta$ (Brdicka, 2000, 73-75), respectively. With regard to Figure 2.3, the shear strains $\varepsilon_{n\varphi}$, $\varepsilon_{n\theta}$ and $\varepsilon_{\varphi n}$, $\varepsilon_{\theta n}$ along the axes x_n and x_φ, x_θ, respectively, have the forms (Ceniga, 2007, 35-38; Ceniga, 2008, 23-26)

$$\varepsilon_{n\varphi} = \tan\left[\angle\left(|12|,|12^*|\right)\right] = \frac{1}{x_n\, d\varphi}\left(\frac{\partial u_n}{\partial \varphi}\, d\varphi\right) = \frac{1}{x_n}\frac{\partial u_n}{\partial \varphi}, \tag{2.3}$$

$$\varepsilon_{n\theta} = \tan\left[\angle\left(|13|,|13^*|\right)\right] = \frac{1}{x_n\, d\theta}\left(\frac{\partial u_n}{\partial \theta}\, d\theta\right) = \frac{1}{x_n}\frac{\partial u_n}{\partial \theta} = \frac{\Theta}{x_n}\frac{\partial u_n}{\partial \nu}, \tag{2.4}$$

where Θ is given by Equation (1.13), and $\varepsilon_{n\varphi} = \varepsilon_{\varphi n}$, $\varepsilon_{n\theta} = \varepsilon_{\theta n}$ (Brdicka, 2000, 68-71). Due to $u_\varphi = u_\theta$, we get $\varepsilon_{\varphi\theta} = \varepsilon_{\theta\varphi} = \propto (\partial u_\varphi/\partial\theta) + (\partial u_\theta/\partial\varphi) = 0$, $\varepsilon_{\varphi\theta}$ and $\varepsilon_{\theta\varphi}$ are shear strains along the axes x_φ, x_θ, respectively.

2.2 Equilibrium Equations

Mechanics of a solid continuum results from the condition of the equilibrium of forces, which acts on sides of an infinitesimal part of a solid continuum. The equilibrium equations of the forces, which act on the sides of the infinitesimal spherical cap are determined with respect to the axes x_n, x_φ, x_θ at the point P (see Figure 2.1). In case of the axis x_n (see Figure 2.4), we get (Ceniga, 2007, 35-38; Ceniga, 2008, 23-26)

$$\left(\sigma_n + \frac{\partial \sigma_n}{\partial x_n}\, dx_n\right)(x_n + dx_n)\, d\varphi\,(x_n + dx_n)\, d\theta$$

$$+\left(\sigma_{n\varphi} + \frac{\partial \sigma_{n\varphi}}{\partial \varphi}\, d\varphi\right)\cos\left(\frac{d\varphi}{2}\right) x_n\, d\theta\, dx_n$$

$$+\left(\sigma_{n\theta} + \frac{\partial \sigma_{n\theta}}{\partial \theta}\, d\theta\right)\cos\left(\frac{d\theta}{2}\right) x_n\, d\varphi\, dx_n$$

$$-\left[\sigma_n\, x_n\, d\varphi\, x_n\, d\theta + \left(\sigma_\varphi + \frac{\partial \sigma_\varphi}{\partial \varphi}\, d\varphi\right)\sin\left(\frac{d\varphi}{2}\right) x_n\, d\theta\, dx_n\right.$$

$$+\left(\sigma_\theta+\frac{\partial\sigma_\theta}{\partial\theta}\,d\theta\right)\sin\left(\frac{d\theta}{2}\right)x_n\,d\varphi\,dx_n+\sigma_\varphi\,\sin\left(\frac{d\varphi}{2}\right)x_n\,d\theta\,dx_n$$

$$+\sigma_\theta\,\sin\left(\frac{d\theta}{2}\right)x_n\,d\varphi\,dx_n+\sigma_{n\varphi}\,\cos\left(\frac{d\varphi}{2}\right)x_n\,d\theta\,dx_n$$

$$+\sigma_{n\theta}\,\cos\left(\frac{d\theta}{2}\right)x_n\,d\varphi\,dx_n\;\Bigg]=0\,, \tag{2.5}$$

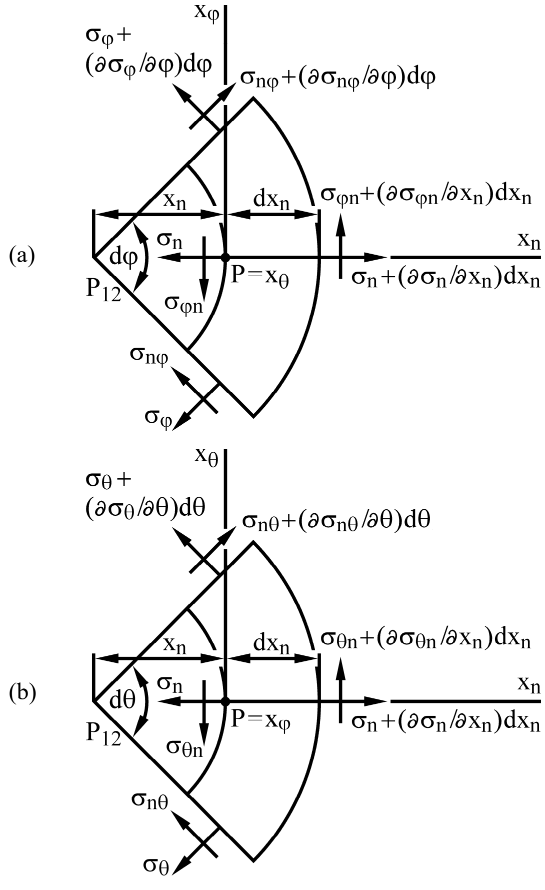


Figure 2.4. The infinitesimal spherical cap in the plane (a) $x_n x_\varphi$, (b) $x_n x_\theta$ (see Figures 1.4, 2.1). The normal stress σ_n, the tangential stresses σ_φ, σ_θ, the shear stresses $\sigma_{n\varphi} = \sigma_{\varphi n}$, $\sigma_{n\theta} = \sigma_{\theta n}$, along with changes of these stresses, acting on the sides of the infinitesimal spherical cap at the point P.

In case of the axis x_φ (see Figure 2.4), we get

$$\left(\sigma_\varphi + \frac{\partial \sigma_\varphi}{\partial \varphi} d\varphi\right)\cos\left(\frac{d\varphi}{2}\right) x_n\, d\theta\, dx_n$$
$$+\left(\sigma_{\varphi n} + \frac{\partial \sigma_{\varphi n}}{\partial x_n} dx_n\right)(x_n + dx_n)d\varphi(x_n + dx_n)d\theta$$
$$+\left(\sigma_{n\varphi} + \frac{\partial \sigma_{n\varphi}}{\partial \varphi} d\varphi\right)\sin\left(\frac{d\varphi}{2}\right) x_n\, d\theta\, dx_n + \sigma_{n\varphi}\sin\left(\frac{d\varphi}{2}\right) x_n\, d\theta\, dx_n$$
$$-\left(\sigma_\varphi \cos\left(\frac{d\varphi}{2}\right) x_n\, d\theta\, dx_n + \sigma_{\varphi n}\, x_n\, d\varphi\, x_n\, d\theta\right) = 0\,, \tag{2.6}$$

In case of the axis x_θ (see Figure 2.4), we get

$$\left(\sigma_\theta + \frac{\partial \sigma_\theta}{\partial \theta} d\theta\right)\cos\left(\frac{d\theta}{2}\right) x_n\, d\varphi\, dx_n$$
$$+\left(\sigma_{\theta n} + \frac{\partial \sigma_{\theta n}}{\partial x_n} dx_n\right)(x_n + dx_n)d\varphi(x_n + dx_n)d\theta$$
$$+\left(\sigma_{n\theta} + \frac{\partial \sigma_{n\theta}}{\partial \theta} d\theta\right)\sin\left(\frac{d\theta}{2}\right) x_n\, d\varphi\, dx_n + \sigma_{n\theta}\sin\left(\frac{d\theta}{2}\right) x_n\, d\varphi\, dx_n$$
$$-\left(\sigma_\theta \cos\left(\frac{d\theta}{2}\right) x_n\, d\varphi\, dx_n + \sigma_{\theta n}\, x_n\, d\varphi\, x_n\, d\theta\right) = 0\,. \tag{2.7}$$

where $|13| = x_n\, d\theta$, $|1'3'| = (u_n + x_n)\, d\theta$ is considered instead of $|13| = x_n \sin\theta\, d\theta$, $|1'3'| = (u_n + x_n) \sin\theta\, d\theta$ (Brdicka, 2000, 73-75), respectively. Due to $d\varphi \approx 0$, $d\theta \approx 0$, $dr \approx 0$, we get $\sin(d\varphi/2)\approx d\varphi/2$, $\sin(d\theta\,/2)\approx d\theta/2$, $\cos(d\varphi/2) = \cos(d\theta/2) \approx 1$, $(dr)^2 = (d\varphi)^2 = (d\theta)^2 = 0$ (Brdicka, 2000, 76-77). Consequently, the equilibrium equations (2.5)-(2.7) are derived as (see Equation (1.13))

$$2\sigma_n - \sigma_\varphi - \sigma_\theta + x_n \frac{\partial \sigma_n}{\partial x_n} + \frac{\partial \sigma_{n\varphi}}{\partial \varphi} + \Theta \frac{\partial \sigma_{n\theta}}{\partial \nu} = 0\,, \tag{2.8}$$

$$\frac{\partial \sigma_{\varphi}}{\partial \varphi} + 3\sigma_{n\varphi} + x_n \frac{\partial \sigma_{n\varphi}}{\partial x_n} = 0, \tag{2.9}$$

$$\Theta \frac{\partial \sigma_{\theta}}{\partial \nu} + 3\sigma_{n\theta} + x_n \frac{\partial \sigma_{n\theta}}{\partial x_n} = 0, \tag{2.10}$$

where σ_n and σ_φ, σ_θ are normal and tangential stress along the axes x_n and x_φ, x_θ, respectively; $\sigma_{n\varphi}$, $\sigma_{n\theta}$ and $\sigma_{\varphi n}$, $\sigma_{\theta n}$ are shear stress along the axes x_n and x_φ, x_θ, respectively. Due to $\varepsilon_{\varphi\theta} = \varepsilon_{\theta\varphi}$, we get $\sigma_{\varphi\theta} = \sigma_{\theta\varphi} = 0$, where $\sigma_{\varphi\theta}$ is a shear stress.

2.3 Hooke Law

With regard to $\varepsilon_{\varphi\theta} = 0$, $\sigma_{\varphi\theta} = 0$, Hooke's law has the form (Brdicka, 2000, 60-62)

$$\varepsilon_n = s_{11}\sigma_n + s_{12}\left(\sigma_\varphi + \sigma_\theta\right), \tag{2.11}$$

$$\varepsilon_\varphi = s_{12}\left(\sigma_n + \sigma_\theta\right) + s_{11}\sigma_\varphi, \tag{2.12}$$

$$\varepsilon_\theta = s_{12}\left(\sigma_n + \sigma_\varphi\right) + s_{11}\sigma_\theta, \tag{2.13}$$

$$\varepsilon_{n\theta} = s_{44}\sigma_{n\theta}, \tag{2.14}$$

$$\varepsilon_{n\varphi} = s_{44}\sigma_{n\varphi}, \tag{2.15}$$

where s_{11}, s_{12} have the form (Brdicka, 2000, 62-63)

$$s_{11} = \frac{1}{E}, \quad s_{12} = -\frac{\mu}{E}, \quad s_{11} = \frac{2(1+\mu)}{E}, \tag{2.16}$$

and E, μ represent Young modulus, Poisson's ratio (Brdicka, 2000, 60-62), respectively. In case of the ellipsoidal inclusion and the matrix, we get $E = E_{IN}$, $\mu = \mu_{IN}$ and $E = E_M$, $\mu = \mu_M$, respectively. With regard to Equations (2.1)-(2.4), (2.11)-(2.15), we get (Ceniga, 2007, 22-23; Ceniga, 2008, 27-28)

$$\sigma_n = (c_1 + c_2)\frac{\partial u_n}{\partial x_n} - 2c_2\frac{u_n}{x_n}, \tag{2.17}$$

$$\sigma_\varphi = \sigma_\theta = -c_2\frac{\partial u_n}{\partial x_n} + c_1\frac{u_n}{x_n}, \tag{2.18}$$

$$\sigma_{n\varphi} = -\frac{1}{s_{44}\,x_n}\,\frac{\partial u_n}{\partial \varphi}, \tag{2.19}$$

$$\sigma_{n\theta} = -\frac{\Theta}{s_{44}\,x_n}\,\frac{\partial u_n}{\partial \nu}, \tag{2.20}$$

where Θ is given by Equation (1.13), and c_1, c_2, c_2 are derived as

$$c_1 = \frac{E}{(1+\mu)(1-2\mu)},\ c_2 = -\frac{\mu E}{(1+\mu)(1-2\mu)},\ c_3 = -4(1-\mu) < 0, \tag{2.21}$$

and $c_3 < 0$ due to $\mu < 0.5$ for real isotropic components (Skocovsky and Bokuvka and Palcek, 1996, 75-79).

If $a_{1i} = \cos[\angle(x_1, x_i)]$ $(i = n, \varphi, \theta)$ represent a direction cosine of an angle formed by axes x_1, x_i, then, with respect to Figures 1.4, 1.5, we get

$$a_{1n} = \cos\varphi\,\sin\theta,\quad a_{1\varphi} = \sin\varphi\,\sin\theta,\quad a_{1\theta} = \cos\theta,$$
$$a_{\varphi 1} = -\sin\varphi,\quad a_{\theta 1} = -\cos\varphi\cos\theta, \tag{2.22}$$

where $\cos\theta$, $\sin\theta$ are given by Equation (1.12). The stress σ_1 along the axis x_1 has the form

$$\sigma_1 = a_{1n}\,\sigma_n + a_{1\varphi}\,\sigma_\varphi + a_{1\theta}\,\sigma_\theta$$
$$+ a_{1n}(\sigma_{n\varphi} + \sigma_{n\theta}) + a_{1\varphi}\,\sigma_{\varphi n} + a_{1\theta}\,\sigma_{\theta n}. \tag{2.23}$$

With regard to Equations (2.17)-(2.20) and due to $\sigma_{n\varphi} = \sigma_{\varphi n}$, $\sigma_{n\theta} = \sigma_{\theta n}$ (Brdicka, 2000, 65-67), we get

$$\sigma_1 = \gamma_1 \frac{\partial u_n}{\partial x_n} + \gamma_2 \frac{u_n}{x_n} + \frac{1}{s_{44}\, x_n}\left(\gamma_3 \frac{\partial u_n}{\partial \varphi} + \gamma_4 \frac{\partial u_n}{\partial \nu} \right), \tag{2.24}$$

where γ_i ($i = 1,\ldots,4$) is derived as

$$\gamma_1 = a_{1n}\left(c_1 + c_2\right) - \left(a_{1\varphi} + a_{1\theta}\right)c_2, \quad \gamma_2 = \left(a_{1\varphi} + a_{1\theta}\right)c_1 + 2a_{1n}\, c_2\,,$$
$$\gamma_3 = a_{1n} + a_{1\varphi}, \quad \gamma_4 = \Theta\left(a_{1\varphi} + a_{1\theta}\right), \tag{2.25}$$

where Θ is given by Equation (1.13). As presented in Chapter 10, the mathematical models of the material micro-strengthening $\sigma_{st} = \sigma_{st}(x_1)$ and the material macro-strengthening $\overline{\sigma_{st}}$ result from the stress σ_1 (see Equations (2.24), (2.25)). If Equations (2.17)-(2.20) are substituted to Equation (2.21) and to ∂Eq.(2.9)/∂φ, then Equations (2.8)-(2.10) are derived as (Ceniga, 2007, 25; Ceniga, 2008, 30)

$$x_n^2 \frac{\partial^2 u_n}{\partial^2 x_n} + 2x_n \frac{\partial u_n}{\partial x_n} - 2u_n + \frac{U_n}{s_{44}\left(c_1 + c_2\right)} = 0\,, \tag{2.26}$$

$$x_n \frac{\partial U_n}{\partial x_n} = c_3\, U_n\,, \tag{2.27}$$

where U_n has the form

$$U_n = \frac{\partial^2 u_n}{\partial \varphi^2} + \Theta^2 \frac{\partial^2 u_n}{\partial \nu^2}\,. \tag{2.28}$$

2.4 Elastic Energy

The system of the differential equations (2.26), (2.28) is solved by the different mathematical procedures in Sections 5.1, 6.1, 7.1, 8.1, 9.1, which result in different mathematical solutions for the thermal and phase-transfor-mation stresses, and the principle of minimum potential energy W_p is con-sidered (Brdicka, 2000, 96-98). Consequently, such a combination of the different mathematical solutions for the matrix and the ellipsoidal inclu-sion is considered to exhibit minimum potential energy $W_p = W_d + W_v + W_s$ (Brdicka, 2000, 96-98), where W_d is deformation energy, W_v and W_s is energy induced by volume and surface forces, respectively. The model system in Figure 1.1 is not acted by the volume and surface forces, i.e., $W_v = W_s = 0$, and then $W_p = W_d$ is induced by the thermal and phase-transformation stresses in the ellipsoidal inclusion and the cell matrix. The sum $W_p = W_d = W_{IN} + W_M$ represents potential energy, which is accumulated in the cubic cell (see Figure 1.2), where W_{IN} and W_M is elastic energy of the ellipsoidal inclusion and the cell matrix, respectively.

The elastic energy density w_q in the cell matrix ($q = M$) and the ellipsoidal inclusion ($q = IN$) is derived as (Brdicka, 2000, 94-95)

$$w_q = \frac{1}{2}\left(\varepsilon_{nq}\,\sigma_{nq} + \varepsilon_{\varphi q}\,\sigma_{\varphi q} + \varepsilon_{\theta q}\,\sigma_{\theta q}\right) + \varepsilon_{n\varphi q}\,\sigma_{n\varphi q} + \varepsilon_{n\theta q}\,\sigma_{n\theta q}\,,$$
$$q = M, IN \tag{2.29}$$

and the elastic energy W_q ($q = M$, IN) has the form

$$W_M = \int_{V_M} w_M\, dV_M = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_{x_{IN}}^{x_M} w_M\, x_n^2\, dx_n\, d\varphi\, \sin\theta\, d\theta$$
$$= 8\int_0^{\pi/2}\int_0^{\pi/2}\int_{x_{IN}}^{x_M} w_M\, x_n^2\, \Omega\, dx_n\, d\varphi\, dv\,,$$
$$W_{IN} = \int_{V_{IN}} w_{IN}\, dV_{IN} = 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{x_{IN}} w_{IN}\, x_n^2\, dx_n\, d\varphi\, \sin\theta\, d\theta$$
$$= 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{x_{IN}} w_{IN}\, x_n^2\, \Omega\, dx_n\, d\varphi\, dv\,, \tag{2.30}$$

where $\sin\theta\, \mathrm{d}\theta$ and Ω are given by Equation (1.13)

CHAPTER 3

REASON FOR STRESSES

The thermal stresses, which originate during a cooling process at the temperature $T \in \langle T_f, T_r \rangle$, result from the condition $\alpha_M \neq \alpha_{IN}$, where T_f is final temperature of the cooling process; T_r is relaxation temperature of a real matrix-inclusion composite material; α_M and α_{IN} represent thermal expansion coefficients of the matrix and the ellipsoidal inclusion, respectively. The phase transformation, which originates at the temperature $T_{tq} \in \langle T_f, T_r \rangle$, results in the strain ε_{tq} ($q = M, IN$), where T_{tM}, ε_{tM} and T_{tIN}, ε_{tIN} are related to the matrix and the ellipsoidal inclusion, respectively. Consequently, the strain ε_{tq} is a reason for the phase-transformation stresses.

If $T \geq T_r$ and $T_{tq} \geq T_r$, then the stresses are relaxed by thermal-activated processes, where $T_r = (0.35\text{-}0.4) \times T_m$ (Skocovsky and Bokuvka and Palcek, 1996, 42-44), T_m is melting temperature of a real matrix-inclusion composite material. If the inclusions are formed in a liquid matrix, then T_m is a minimum of the set $\{T_{mIN}, T_{mM}\}$, where T_{mIN} and T_{mM} is melting temperature of the inclusion and the matrix, respectively. If the inclusions are formed in a solid matrix with the melting temperature T_{mM}, then we get $T_m = T_{mM}$. If the matrix-inclusion composite consists of two types of crystal grains, then T_m represents melting temperature of this material.

If $T_{tq} \in \langle T_f, T_r \rangle$, then the coefficient $\beta_q = \beta_q(T)$ at $T \in \langle T_f, T_{tq} \rangle \subset \langle T_f, T_r \rangle$ is derived as (Ceniga, 2007, 34; Ceniga, 2008, 22)

$$\beta_q = \varepsilon_{tq} + \int_{T_{tq}}^{T_r} \alpha_q^{(1)}\, dT + \int_{T}^{T_{tq}} \alpha_q^{(2)}\, dT \,,$$

$$T_{tq} \in \left\langle T_f, T_r \right\rangle, \quad T \in \left\langle T_f, T_{tq} \right\rangle \subset \left\langle T_f, T_r \right\rangle, \quad q = M, IN \,, \tag{3.1}$$

where $\alpha_q^{(1)} = \alpha_q^{(1)}(T)$ and $\alpha_q^{(2)} = \alpha_q^{(2)}(T)$ are related to $T \geq T_{tq}$ and $T \leq T_{tq}$, respectively, and ε_{tq} is given by Equations (3.17), (3.18). If $\alpha_q^{(1)}$, $\alpha_q^{(2)}$ are not functions of $T \geq T_{tq}$, $T \leq T_{tq}$, respectively, then we get

$$\beta_q = \varepsilon_{tq} + \alpha_q^{(1)}\left(T_r - T_{tq}\right) + \alpha_q^{(2)}\left(T_{tq} - T\right),$$
$$T_{tq} \in \left\langle T_f, T_r \right\rangle, \quad T \in \left\langle T_f, T_{tq} \right\rangle \subset \left\langle T_f, T_r \right\rangle, \quad q = M, IN, \tag{3.2}$$

If $T > T_{tq}$, then $\beta_q = \beta_q(T)$ at $T \in \langle T_{tq}\ T_r \rangle \subset \langle T_f, T_r \rangle$ has the form

$$\beta_q = \int_T^{T_r} \alpha_q^{(1)}\, dT,$$
$$T_{tq} \in \left\langle T_f, T_r \right\rangle, \quad T \in \left\langle T_{tq}, T_r \right\rangle \subset \left\langle T_f, T_r \right\rangle, \quad q = M, IN, \tag{3.3}$$

If $\alpha_q^{(1)} \neq \alpha_q^{(1)}(T)$ for $T \geq T_{tq}$, then we get

$$\beta_q = \alpha_q^{(1)}\left(T_r - T\right), \quad T \in \left\langle T_{tq}, T_r \right\rangle \subset \left\langle T_f, T_r \right\rangle, \quad q = M, IN. \tag{3.4}$$

If $T_{tq} \notin \left\langle T_f, T_r \right\rangle$, then $\beta_q = \beta_q(T)$ at $T \in \langle T_f, T_r \rangle$ is derived as

$$\beta_q = \int_T^{T_r} \alpha_q\, dT, \quad T_{tq} \notin \left\langle T_f, T_r \right\rangle, \quad T \notin \left\langle T_f, T_r \right\rangle, \quad q = M, IN, \tag{3.5}$$

where $\alpha_q = \alpha_q(T)$ is related to the condition $T_{tq} \notin \left\langle T_f, T_r \right\rangle$. If $\alpha_q \neq \alpha_q(T)$ for $T \in \langle T_f, T_r \rangle$, then we get

$$\beta_q = \alpha_q\left(T_r - T\right), \quad T_{tq} \notin \left\langle T_f, T_r \right\rangle, \quad T \in \left\langle T_f, T_r \right\rangle, \quad q = M, IN. \tag{3.6}$$

Isotropic material components are characterized by cubic crystalline latices (CCL), which exhibit the following modifications (Skocovsky and Bokuvka and Palcek, 1996, 15-18)

- the simple modification ($K6$): one atom at each corner point of CCL,
- the body-centered modification ($K8$): one atom at the center of CCL, one atom at each corner point of CCL,
- the face-centered modification ($K12$): one central atom on each side of CCL, one atom at each corner point of CCL.

The phase transformation of CCL in the matrix ($q = M$) and/or the ellipsoidal inclusion ($q = IN$) at the temperature $T_{tq} \in \langle T_f, T_r \rangle$ represents the transformation $a_{qI} \rightarrow a_{qII}$ at $T = T_{tq}$, where a_{qI}, a_{qII} are dimensions of CCL at $T \geq T_{tq}$, $T \leq T_{tq}$, respectively, and $a_{qm} \subset \{a_{qm}^{(K6)}, a_{qm}^{(K8)}, a_{qm}^{(K12)}\}$ ($m = I, II$). The transformation $a_{qI} \rightarrow a_{qII}$ results in the strain ε_{tq}, which induces the phase-transformation stresses.

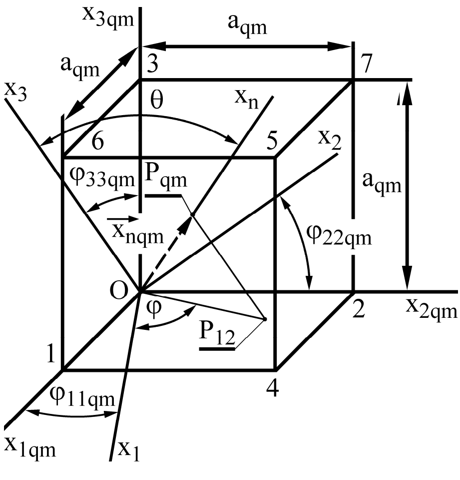


Figure 3.1. The cubic crystalline lattice (CCL) in the matrix ($q = M$) and the ellipsoidal inclusion ($q = IN$); the dimension a_{qm} along the axis x_{iqm} ($i = 1,2,3$) of the Cartesian system ($Ox_{1qm}x_{2qm}x_{3qm}$). The angle $\varphi_{ijqm} = \angle(x_{iqm}, x_j)$ defines a position of CCL with respect to the Cartesian system ($Ox_1x_2x_3$) (see Figure 1.2). As an example, the angles φ_{11qm}, φ_{22qm}, φ_{33qm} are shown. P_{qm} is an intersection point of the normal x_n and the surface *1456*. P_{12} is a projection of P_{qm} onto the plane x_1x_2; the vector $\vec{x}_{nqm} = \vec{OP}_{qm}$.

Figure 3.1 shows CCL with the dimension a_{qm} along the axis x_{iqm} of the Cartesian system ($Ox_{1qm}x_{2qm}x_{3qm}$) ($i = 1,2,3$). The angle $\varphi_{ijqm} = \angle(x_{iqm}, x_j)$ (see Figure 3.1), which is formed by the axes x_{iqm}, x_j ($i, j = 1,2,3$; $m = I, II$), defines a position of CCL with respect to the Cartesian system ($Ox_1x_2x_3$).

As an example, the angles φ_{11qm}, φ_{22qm}, φ_{33qm} are shown in Figure 3.1. The coefficient a_{ijqm}, which represents a direction cosine of φ_{ijqm}, is derived as (Ceniga, 2007, 35-38; Ceniga, 2008, 23-26)

$$a_{ijqm} = \cos\varphi_{ijqm} = \cos\left[\angle\left(x_{iqm}, x_j\right)\right],$$
$$i, j = 1,2,3;\ q = M, IN;\ m = I, II\ . \tag{3.7}$$

As shown in Figure 3.1, P_{qm} is an intersection point of the normal x_n with one of the surfaces *1456*, *2754*, *3657*, and P_{12} is a projection of P_{qm} onto the plane x_1x_2. The length $|\vec{x}_{nqm}| = |\vec{OP}_{qm}|$ of the vector $\vec{x}_{nqm} = \vec{OP}_{qm}$ along the axis x_n in CCL is determined by a_{qm}, φ, θ. The point P_{qm} is defined by the coordinates (x_1,x_2,x_3) in the Cartesian system $(Ox_1x_2x_3)$ or by $(x_{1qm},x_{2qm},x_{3qm})$ in the Cartesian system $(Ox_{1qm}x_{2qm}x_{3qm})$, and then we get (Ceniga, 2007, 35-38; Ceniga, 2008, 23-26)

$$x_{iqm} = \sum_{j=1}^{3} a_{ijqm}\, x_j, \quad i = 1,2,3; \quad q = M, IN; \quad m = I, II\ , \tag{3.8}$$

where $\sum_{i=1}^{3}\left(x_{iqm}\right)^2 = \sum_{i=1}^{3}\left(x_i\right)^2$, $\sum_{i=1}^{3} a_{ijqm}\, a_{ikqm} = \delta_{jk}$ $(j,k = 1,2,3)$, and δ_{jk} is Kronecker's delta, i.e., $\delta_{jk} = 0$ and $\delta_{jk} = 1$ for $j \neq k$ and $j = k$ (Rektorys, 1973, 143), respectively. The unit vector $\vec{e}_n$, which is derived in $(Ox_1x_2x_3)$, is derived as (Ceniga, 2007, 35-38; Ceniga, 2008, 23-26)

$$\vec{e}_n = \sum_{i=1}^{3} a_{iqm}^{(n)}\, \vec{e}_{iqm}, \quad a_{iqm}^{(n)} = \cos\left[\angle\left(x_{iqm}, x_n\right)\right] = \sum_{j=1}^{3} a_{nj}\, a_{ijqm} \quad ,$$
$$i = 1,2,3; \quad q = M, IN; \quad m = I, II\ , \tag{3.9}$$

If P_{qm} with the coordinates $(Ox_{1qm}x_{2qm}x_{3qm})$, is a point of the surface *1456*, i.e., $P_{qm} \subset 1456$, then the length $|\vec{x}_{nqm}| = |\vec{OP}_{qm}|$ of the vector $\vec{x}_{nqm} = \vec{OP}_{qm}$ long the axis x_n in CCL has the form (Ceniga, 2007, 35-38; Ceniga, 2008, 23-26)

$$\left| \overrightarrow{x_{nqm}} \right| = \frac{a_{qm}}{a_{1qm}^{(n)}}, \quad q = M, IN; \quad m = I, II, \tag{3.10}$$

where $x_{nqm} = a_{qm}$, $x_{\varphi qm} = a_{qm}\, a_{2qm}^{(n)} / a_{1qm}^{(n)} \le a_{qm}$, $x_{vqm} = a_{qm}\, a_{3qm}^{(n)} / a_{1qm}^{(n)} \le a_{qm}$. The surface *1456* with the normal x_n is determined by each of the conditions (Ceniga, 2007, 35-38; Ceniga, 2008, 23-26)

$$\frac{a_{2qm}^{(n)}}{a_{1qm}^{(n)}} \le 1 \quad \wedge \quad \frac{a_{3qm}^{(n)}}{a_{1qm}^{(n)}} \le 1, \quad q = M, IN; \quad m = I, II. \tag{3.11}$$

If $P_{qm} \subset$ *2754*, then we get (Ceniga, 2007, 35-38; Ceniga, 2008, 23-26)

$$\left| \overrightarrow{x_{nqm}} \right| = \frac{a_{qm}}{a_{2qm}^{(n)}}, \quad q = M, IN; \quad m = I, II, \tag{3.12}$$

where $x_{nqm} = a_{qm}\, a_{1qm}^{(n)} / a_{2qm}^{(n)} \le a_{qm}$, $x_{\varphi qm} = a_{qm}$ $x_{vqm} = a_{qm}\, a_{3qm}^{(n)} / a_{2qm}^{(n)} \le a_{qm}$. The surface *2754* with the normal x_n is determined by each of the conditions (Ceniga, 2007, 35-38; Ceniga, 2008, 23-26)

$$\frac{a_{1qm}^{(n)}}{a_{2qm}^{(n)}} \le 1 \quad \wedge \quad \frac{a_{3qm}^{(n)}}{a_{2qm}^{(n)}} \le 1, \quad q = M, IN; \quad m = I, II. \tag{3.13}$$

If $P_{qm} \subset$ *3657*, then we get (Ceniga, 2007, 35-38; Ceniga, 2008, 23-26)

$$\left| \overrightarrow{x_{nqm}} \right| = \frac{a_{qm}}{a_{3qm}^{(n)}}, \quad q = M, IN; \quad m = I, II, \tag{3.14}$$

where $x_{nqm} = a_{qm}\, a_{1qm}^{(n)} / a_{3qm}^{(n)} \le a_{qm}$, $x_{\varphi qm} = a_{qm}\, a_{2qm}^{(n)} / a_{3qm}^{(n)} \le a_{qm}$ and $x_{vqm} = a_{qm}$. The surface *3657* with the normal x_n is determined by each of the conditions (Ceniga, 2007, 35-38; Ceniga, 2008, 23-26)

$$\frac{a_{1qm}^{(n)}}{a_{3qm}^{(n)}} \le 1 \quad \wedge \quad \frac{a_{2qm}^{(n)}}{a_{3qm}^{(n)}} \le 1, \quad q = M, IN; \quad m = I, II\,. \tag{3.15}$$

The surface with the normal x_n is determined by each of the conditions

$$\frac{a_{jqm}^{(n)}}{a_{iqm}^{(n)}} \le 1 \quad \wedge \quad \frac{a_{kqm}^{(n)}}{a_{iqm}^{(n)}} \le 1\,,$$
$$i, j, k - 1,2,3; \quad i \ne j \ne k; \quad q = M, IN; \quad m = I, II\,. \tag{3.16}$$

The strain ε_{tq} ($q = M,IN$) has the form

$$\varepsilon_{tq} = \frac{\left|\overrightarrow{x}_{nqII}\right| - \left|\overrightarrow{x}_{nqI}\right|}{\left|\overrightarrow{x}_{nqII}\right|}, \quad q = M, IN\,, \tag{3.17}$$

where $\left|\overrightarrow{x}_{nqI}\right|$, $\left|\overrightarrow{x}_{nqII}\right|$ are related to the temperature $T = T_{tq}$. If $T_{tq} \in \langle T_f, T_{tq} \rangle$, then $\left|\overrightarrow{x}_{nqII}\right|$ in Equation (3.16) is replaced by the following formula

$$\left|\overrightarrow{x}_{nqII}\right|_T = \left|\overrightarrow{x}_{nqII}\right| \left(1 - \int_{T_f}^{T_{tq}} \alpha_q^{(2)}\, dT \right),$$
$$T \in \left\langle T_f, T_{tq} \right\rangle, \quad q = M, IN\,. \tag{3.18}$$

CHAPTER 4

MECHNICAL CONSTRAINTS

The mechanical constraints of the model system in Figure 1.2 at the temperature $T \in \langle T_f, T_r \rangle$ are determined for

- the matrix-inclusion boundary with respect to the condition $\beta_{IN} \neq \beta_M$ (see Chapter 3), which is a reason of the normal stress p_n, act-ing at the matrix-inclusion boundary,
- the cell surface with respect to the displacement u_{nM}.

The mechanical constraints are described by corresponding mathema-tical conditions, which determine integration constants in mathematical solutions of the system of differential equations (2.26), (2.27). Using Cramer's rule (see Section 12.2), the integration constants are determined by the mathematical boundary conditions (4.1)-(4.5).

The difference β_M - $\beta_{IN} \neq 0$ results in the normal displacements $(u_{nM})_{xn=xIN} \neq 0$, $(u_{nIN})_{xn=xIN} \neq 0$ at the matrix-inclusion boundary where $(u_{nM})_{xn=xIN}$ and $(u_{nIN})_{xn=xIN}$ are a reason of the stresses in the cell matrix and the ellipsoidal inclusion, respectively.

4.1 Cell matrix

The absolute values $|u_{nM}|$, $|\varepsilon_M|$, $|\sigma_M|$ (see Equations (2.1)-(2.4), (2.17)-(2.20)) are required to exhibit decreasing functions of the variable $x_n \in \langle x_{IN}, x_M \rangle$, with maximum values at the matrix-inclusion boun-dary, i.e., for $x_n = x_{IN}$, where x_{IN}, x_M are given by Equation (1.15). These decreasing functions result from the following mandatory conditions (Ceniga, 2007, 67; Ceniga, 2008, 51)

$$(\sigma_{nM})_{x_n=x_{IN}} = -p_n, \tag{4.1}$$

$$\left(u_{nM}\right)_{x_n=x_M} = 0\,, \tag{4.2}$$

where the normal stress p_n, acting at the matrix-inclusion boundary, is given by Equation (4.6). Equations (4.1) and (4.2) represent stress and geometric boundary conditions, respectively.

As mentioned above, the displacement u_{nM} is a consequence of the difference β_M - $\beta_{IN} \neq 0$, and does not result from the dimension change $\Delta d = d\, \alpha_M\, \Delta T$. If $\left(u_{nM}\right)_{x_n=x_M} > 0$ or $\left(u_{nM}\right)_{x_n=x_M} < 0$, then $\Delta d > 0$ or $\Delta d < 0$ at the constant temperature $T\in\langle T_f,\ T_r\rangle$, respectively, and this increase or decrease of the cell dimension d is physically unacceptable.

The point P_2 on the cell surface (see Figure 1.5) is related to two neighbouring cubic cell. Let $u_{nMA} = u_{nMA}(x_{nA},\varphi_A,\nu_A)$ represent a function of the variables x_{nA}, φ_A, ν_A in a certain cell, e.g., in the cell in Figure 1.5, and $u_{nMB} = u_{nMB}(x_{nB},\varphi_B,\nu_B)$ represent a function of x_{nB}, φ_B, ν_B in a neigbhouring cell. Let $u_{nMA} = u_{nMA}(x_{nA},\varphi_A,\nu_A)$ and $u_{nMB} = u_{nMB}(x_{nB},\varphi_B,\nu_B)$ are connected at the point P_2. The model system is imaginary divided into identical cubic cells (see Figure 1.2), and then the cell surface is not a physical boundary.

This connection is assumed to be 'smooth', and then u_{nMA} and u_{nMB} are assumed not to create a singular connection at the point P_2, which is assumed not to represent a singular point. Due to this non-singularity assumption, the function $u_{nM} = u_{nM}(x_n,\varphi,\nu)$ of the variable $x_n\in\langle x_{IN},x_M\rangle$ is assumed to be extremal on the cell surface, i.e., for $x_n = x_M$. The absolute values $|u_{nM}|$ represents a decreasing functions of $x_n\in\langle x_{IN},x_M\rangle$, and then this extreme for $x_n = x_M$ is a minimum of $u_{nM} = u_{nM}(x_n,\varphi,\nu)$. With regard to Equation (2.1), the following condition

$$\left(\varepsilon_{nM}\right)_{x_n=x_M} = \left(\frac{\partial u_{nM}}{\partial x_n}\right)_{x_n=x_M} = 0\,. \tag{4.3}$$

4.2 Ellipsoidal Inclusion

The absolute values $|u_{nIN}|$, $|\varepsilon_{IN}|$, $|\sigma_{IN}|$ (see Equations (2.1)-(2.4), (2.17)-(2.20)) are required to exhibit increasing functions of the variable $x_n\in\langle x_{IN},x_M\rangle$, with maximum values at the matrix-inclusion boundary, i.e., for $x_n = x_{IN}$, and the conditions $(u_{nIN})_{xn=0} \neq \pm\infty$, $(\varepsilon_{nIN})_{xn=0} \neq \pm\infty$, $(\sigma_{nIN})_{xn=0} \neq \pm\infty$ are required to be valid. The boundary conditions are derived as (Ceniga, 2007, 67; Ceniga, 2008, 51)

$$(\sigma_{nIN})_{x_n=x_{IN}} = -p_n, \tag{4.4}$$

$$(u_{nIN})_{x_n=0} = 0, \tag{4.5}$$

where Equations (4.4) and (4.5) represent stress and geometric boundary conditions, respectively.

4.3 Normal Stress

If $\beta_M > \beta_{IN}$ or $\beta_M < \beta_{IN}$, then the normal stress $p_n > 0$ or $p_n < 0$ is compressive or tensile, respectively. If $p_n > 0$ is a compressive normal stress, acting at the matrix-inclusion boundary, then we get $(u_{nM})_{x_n=x_{IN}} = -x_n\,\beta_M$, $(u_{nIN})_{x_n=x_{IN}} = x_n\,\beta_{IN}$. With regard to Equation (2.2), we get $(\varepsilon_{\varphi M})_{x_n=x_{IN}} = -\beta_M$, $(\varepsilon_{\varphi IN})_{x_n=x_{IN}} = \beta_{IN}$. If $(\varepsilon_{\varphi M})_{x_n=x_{IN}} = -p_n\rho_M$, $(\varepsilon_{\varphi IN})_{xn=xIN} = -p_n\rho_{IN}$, then the normal stress has the form

$$p_n = \frac{\beta_M - \beta_{IN}}{\rho_M + \rho_{IN}}, \tag{4.6}$$

where β_q ($q = IN,M$) is determined in Chapter 3. The normal stress p_n is included in formulae for the stresses and strains, where its compressive influence is denoted by the term '$-p_n$'. The coefficients ρ_M and ρ_{IN} are given by Equations (5.26), (6.26), (6.37), (6.48), (6.59), (7.25), (7.36), (7.47), (7.58), (8.24), (8.35), (8.46), (8.57), (9.23), (9.34), (9.45), (9.56), (9.67), (9.78) and (5.27), (8.66), respectively. Consequently, such a combination of ρ_M, ρ_{IN} is considered to result in minimum potential energy $W_p = W_{IN} + W_M$ (see Equation (2.30)).

CHAPTER 5

MATHEMATICAL MODEL 1

5.1 Mathematical procedure 1

If the mathematical procedure $x_n[\partial \text{Eq.}(2.27)/\partial x_n]$ is performed, then we get

$$x_n^2 \frac{\partial^2 U_n}{\partial x_n^2} + \left(1 - c_3\right) x_n \frac{\partial U_n}{\partial x_n} = 0\,, \tag{5.1}$$

where $c_3 < 0$, $U_n = U_n\ (x_n, \varphi, \theta)$ are given by Equation (2.21),(2.28), respectively. If Equation (2.27) is substituted to Equation (5.1), then we get

$$x_n^2 \frac{\partial^2 U_n}{\partial x_n^2} + c_3\left(1 - c_3\right) U_n = 0\,. \tag{5.2}$$

If $U_n = x_n^{\lambda}$, then, with respect to Equation (5.2), we get

$$U_n = C_1\, x_n^{\lambda_1} + C_2\, x_n^{\lambda_2}\,, \tag{5.3}$$

where the integration constants C_1, C_2 are determined by the boundary conditions in Chapter 4. The exponents λ_1, λ_2 have the forms

$$\lambda_1 = \frac{1}{2}\left[1 + \sqrt{1 + 16\left(1 - \mu\right)\left[1 + 4\left(1 - \mu\right)\right]}\right] > 3\,,$$
$$\lambda_2 = \frac{1}{2}\left[1 - \sqrt{1 + 16\left(1 - \mu\right)\left[1 + 4\left(1 - \mu\right)\right]}\right] > -2\,, \tag{5.4}$$

where $\mu < 0.5$ for a real isotropic material (Skocovsky and Bokuvka and Palcek, 1996, 75-79). If Equation (5.3) is substituted to Equation (2.26), we get

$$x_n^2 \frac{\partial^2 u_n}{\partial x_n^2} + 2\left(x_n \frac{\partial u_n}{\partial x_n} - u_n \right) = C_1 x_n^{\lambda_1} + C_2 x_n^{\lambda_2} . \tag{5.5}$$

Using Wronskian's method in Section 12.1 (Rektorys, 1973, 225-227), the mathemati-cal solution of Equation (5.5) has the form

$$u_n = C_1 x_n^{\lambda_1} + C_2 x_n^{\lambda_2} . \tag{5.6}$$

and with regard to Equation (2.1)-(2.4), (2.17)- (2.20), (2.29), (5.6), we get

$$\varepsilon_n = C_1 \lambda_1 x_n^{\lambda_1 - 1} + C_2 \lambda_2 x_n^{\lambda_2 - 1} , \tag{5.7}$$

$$\varepsilon_\varphi = \varepsilon_\theta = C_1 x_n^{\lambda_1 - 1} + C_2 x_n^{\lambda_2 - 1} , \tag{5.8}$$

$$\varepsilon_{n\varphi} = s_{44} \sigma_{n\varphi} = x_n^{\lambda_1 - 1} \frac{\partial C_1}{\partial \varphi} + x_n^{\lambda_2 - 1} \frac{\partial C_2}{\partial \varphi} , \tag{5.9}$$

$$\varepsilon_{n\theta} = s_{44} \sigma_{n\theta} = \Theta \left(x_n^{\lambda_1 - 1} \frac{\partial C_1}{\partial \nu} + x_n^{\lambda_2 - 1} \frac{\partial C_2}{\partial \nu} \right) , \tag{5.10}$$

$$\sigma_n = C_1 \xi_1 x_n^{\lambda_1 - 1} + C_2 \xi_2 x_n^{\lambda_2 - 1} , \tag{5.11}$$

$$\sigma_\varphi = \sigma_\theta = C_1 \xi_3 x_n^{\lambda_1 - 1} + C_2 \xi_4 x_n^{\lambda_2 - 1} , \tag{4.12}$$

$$\sigma_1 = \eta_1 x_n^{\lambda_1 - 1} + \eta_2 x_n^{\lambda_2 - 1} , \tag{5.13}$$

$$w = \kappa_1 x_n^{2(\lambda_1 - 1)} + \kappa_2 x_n^{2(\lambda_2 - 1)} + \kappa_3 x_n^{\lambda_1 + \lambda_2 - 2} , \tag{5.14}$$

where Θ, s_{44} are given by Equations (1.13), (2.16), and ξ_{i}, $\xi_{2+\mathrm{i}}$, $\xi_{2+\mathrm{i}+2\mathrm{j}}$, η_i, κ_j (i = 1,2; j = 1,2,3) have the forms

$$\xi_i = \frac{E[\lambda_i (1 - \mu) + 2\mu]}{(1 - \mu)(1 - 2\mu)}, \quad \xi_{2+i} = \frac{E(1 + \lambda_i \mu)}{(1 - \mu)(1 - 2\mu)} ,$$

$$\xi_{2+i+2j} = \frac{E\{\lambda_i \lfloor \lambda_j (1-\mu) + 4\mu \rfloor + 2\}}{2(1-\mu)(1-2\mu)},$$

$$\eta_i = C_i (\lambda_i \gamma_1 + \gamma_2) + \frac{1}{s_{44}} \left(\gamma_3 \frac{\partial C_i}{\partial \varphi} + \gamma_4 \frac{\partial C_i}{\partial \nu} \right),$$

$$\kappa_i = C_i^2 \xi_{2+3i} + \frac{1}{s_{44}} \left[\left(\frac{\partial C_i}{\partial \varphi} \right)^2 + \Theta^2 \left(\frac{\partial C_i}{\partial \nu} \right)^2 \right],$$

$$\kappa_3 = C_1 C_2 (\xi_6 + \xi_7) + \frac{1}{s_{44}} \left(\frac{\partial C_1}{\partial \varphi} \frac{\partial C_2}{\partial \varphi} + \Theta^2 \frac{\partial C_1}{\partial \nu} \frac{\partial C_2}{\partial \nu} \right), \quad i, j = 1,2. \tag{5.15}$$

5.2 Cell Matrix

With regard to Equations (2.30), (4.1), (4.2), (5.7)-(5.14), we get

$$\varepsilon_{nM} = -p_n \left[\frac{\lambda_{1M}}{\zeta_1} \left(\frac{x_n}{x_M} \right)^{\lambda_{1M}-1} + \frac{\lambda_{2M}}{\zeta_2} \left(\frac{x_n}{x_M} \right)^{\lambda_{2M}-1} \right], \tag{5.16}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = \frac{u_{nM}}{x_n} = -p_n \left[\frac{1}{\zeta_1} \left(\frac{x_n}{x_M} \right)^{\lambda_{1M}-1} + \frac{1}{\zeta_2} \left(\frac{x_n}{x_M} \right)^{\lambda_{2M}-1} \right], \tag{5.17}$$

$$\varepsilon_{n\varphi M} = s_{44M}\, \sigma_{n\varphi M} = -\frac{\partial}{\partial \varphi} \left(\frac{p_n}{\zeta_1 x_M^{\lambda_{1M}-1}} \right) \left(\frac{x_n}{x_M} \right)^{\lambda_{1M}-1} - \frac{\partial}{\partial \varphi} \left(\frac{p_n}{\zeta_2 x_M^{\lambda_{2M}-1}} \right) \left(\frac{x_n}{x_M} \right)^{\lambda_{2M}-1}, \tag{5.18}$$

$$\varepsilon_{n\theta M} = s_{44M}\, \sigma_{n\theta M} = -\Theta^2 \left[\frac{\partial}{\partial \nu} \left(\frac{p_n}{\zeta_1 x_M^{\lambda_{1M}-1}} \right) \left(\frac{x_n}{x_M} \right)^{\lambda_{1M}-1} - \frac{\partial}{\partial \nu} \left(\frac{p_n}{\zeta_2 x_M^{\lambda_{2M}-1}} \right) \left(\frac{x_n}{x_M} \right)^{\lambda_{2M}-1} \right], \tag{5.19}$$

$$\sigma_{nM} = -p_n\left[\frac{\xi_{1M}}{\zeta_1}\left(\frac{x_n}{x_M}\right)^{\lambda_{1M}-1} + \frac{\xi_{2M}}{\zeta_2}\left(\frac{x_n}{x_M}\right)^{\lambda_{2M}-1}\right], \tag{5.20}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -p_n\left[\frac{\xi_{3M}}{\zeta_1}\left(\frac{x_n}{x_M}\right)^{\lambda_{1M}-1} + \frac{\xi_{4M}}{\zeta_2}\left(\frac{x_n}{x_M}\right)^{\lambda_{2M}-1}\right], \tag{5.21}$$

$$\sigma_{1M} = \eta_{1M}\, x_n^{\lambda_{1M}-1} + \eta_{2M}\, x_n^{\lambda_{2M}-1}, \tag{5.22}$$

$$w_M = \kappa_{1M}\, x_n^{2(\lambda_{1M}-1)} + \kappa_{2M}\, x_n^{2(\lambda_{2M}-1)} + \kappa_{3M}\, x_n^{\lambda_{1M}+\lambda_{2M}-2}, \tag{5.23}$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\kappa_{1M}\left(x_M^{2\lambda_{1M}+1} - x_{IN}^{2\lambda_{1M}+1}\right)}{2\lambda_{1M}+1} + \frac{\kappa_{2M}\left(x_M^{2\lambda_{2M}+1} - x_{IN}^{2\lambda_{2M}+1}\right)}{2\lambda_{2M}+1}\right.$$
$$\left. + \frac{\kappa_{3M}\left(x_M^{\lambda_{1M}+\lambda_{2M}+1} - x_{IN}^{\lambda_{1M}+\lambda_{2M}+1}\right)}{\lambda_{1M}+\lambda_{2M}+1}\right]\Omega\, d\varphi\, d\nu, \tag{5.24}$$

where Θ, Ω; x_{IN}, x_M; s_{44M}, λ_{iM}, ξ_{jM} ($i = 1,2$; $j = 1,\ldots,8$) are given by Equations (1.13); (1.15); (2.16), (5.4), (5.15), ζ_i, η_{iM}, κ_{jM} ($i = 1,2$; $j = 1,2,3$; see Equation (5.15)) have the forms

$$\zeta_i = \xi_{iM}\left(\frac{x_{IN}}{x_M}\right)^{\lambda_{iM}-1} - \xi_{3-iM}\left(\frac{x_{IN}}{x_M}\right)^{\lambda_{3-iM}-1},$$

$$\eta_{iM} = -\frac{p_n\left(\lambda_{iM}\gamma_{1M} + \gamma_{2M}\right)}{\zeta_i\, x_n^{\lambda_{iM}-1}} - \frac{\gamma_{3M}}{s_{44M}}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_i\, x_n^{\lambda_{iM}-1}}\right)$$
$$-\frac{\gamma_{4M}}{s_{44M}}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta_i\, x_n^{\lambda_{iM}-1}}\right),$$

$$\kappa_{iM} = \xi_{2+3iM}\left(\frac{p_n}{\zeta_i\, x_n^{\lambda_{iM}-1}}\right)^2 + \frac{1}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta_i\, x_n^{\lambda_{iM}-1}}\right)\right]^2$$

$$+\frac{\Theta^2}{s_{44M}}\left[\frac{\partial}{\partial \nu}\left(\frac{p_n}{\zeta_i\, x_n^{\lambda_{iM}-1}}\right)\right]^2,$$

$$\kappa_{3M}=\frac{p_n^2\left(\xi_{6M}+\xi_{7M}\right)}{\zeta_1\zeta_2\, x_n^{\lambda_{1M}+\lambda_{2M}-2}}+\frac{1}{s_{44M}}\frac{\partial}{\partial \varphi}\left(\frac{p_n}{\zeta_1\, x_n^{\lambda_{1M}-1}}\right)\frac{\partial}{\partial \varphi}\left(\frac{p_n}{\zeta_2\, x_n^{\lambda_{2M}-1}}\right)$$

$$+\frac{\Theta^2}{s_{44M}}\frac{\partial}{\partial \nu}\left(\frac{p_n}{\zeta_1\, x_n^{\lambda_{1M}-1}}\right)\frac{\partial}{\partial \nu}\left(\frac{p_n}{\zeta_2\, x_n^{\lambda_{2M}-1}}\right),\quad i=1,2\,. \tag{5.25}$$

The normal stress p_{n} is given by Equation (4.6). With regard to Equa-tion (5.17), the coefficient ρ_{M} in Equation (4.6) is derived as

$$\rho_M=\frac{(1+\mu_M)(1-2\mu_M)}{E_M}\left[\frac{1}{\lambda_{1M}(1-\mu_M)+2\mu_M}+\frac{1}{\lambda_{2M}(1-\mu_M)+2\mu_M}\right]$$

.

$$\tag{5.26}$$

5.3 Elipsoidal Inclusion

Due to $\lambda_{2IN} < 0$, we get $C_{2IN} = 0$, otherwise $(u_{nIN})_{x_n=0}=\pm\infty$, $(\varepsilon_{nIN})_{x_n=0}=\pm\infty$, $(\sigma_{nIN})_{x_n=0}=\pm\infty$. With regard to Equations (2.30), (4.4), (4.5), (5.7)-(5.14), we get

$$\varepsilon_{nIN}=-\frac{p_n\,\lambda_{1IN}}{\xi_{1IN}}\left(\frac{x_n}{x_{IN}}\right)^{\lambda_{1IN}-1}, \tag{5.27}$$

$$\varepsilon_{\varphi IN}=\varepsilon_{\theta IN}=\frac{u_{nIN}}{x_n}=-\frac{p_n}{\xi_{1IN}}\left(\frac{x_n}{x_{IN}}\right)^{\lambda_{1IN}-1}, \tag{5.28}$$

$$\varepsilon_{n\varphi IN}=s_{44M}\,\sigma_{n\varphi IN}=-x_n^{\lambda_{1IN}-1}\frac{\partial}{\partial \varphi}\left(\frac{p_n}{\xi_{1IN}\,x_{IN}^{\lambda_{1IN}-1}}\right), \tag{5.29}$$

$$\varepsilon_{n\theta IN}=s_{44M}\,\sigma_{n\theta IN}=-\Theta^2\,x_n^{\lambda_{1IN}-1}\frac{\partial}{\partial \nu}\left(\frac{p_n}{\xi_{1IN}\,x_{IN}^{\lambda_{1IN}-1}}\right), \tag{5.30}$$

$$\sigma_{nIN} = -p_n \left(\frac{x_n}{x_{IN}} \right)^{\lambda_{1IN}-1}, \tag{5.31}$$

$$\sigma_{\varphi IN} = \sigma_{\theta IN} = -\frac{p_n \, \xi_{3IN}}{\xi_{1IN}} \left(\frac{x_n}{x_{IN}} \right)^{\lambda_{1IN}-1}, \tag{5.32}$$

$$\sigma_{1IN} = \eta_{1IN} \, x_n^{\lambda_{1IN}-1}, \tag{5.33}$$

$$w_{IN} = \kappa_{1IN} \, x_n^{2(\lambda_{1IN}-1)}, \tag{5.34}$$

$$W_{IN} = \frac{4}{2\lambda_{1IN}+1} \int_0^{\pi/2} \int_0^{\pi/2} \kappa_{1IN} \, x_{IN}^{2\lambda_{1IN}+1} \, \Omega \, d\varphi \, d\nu \,, \tag{5.35}$$

where Θ, Ω; x_{IN}, $s_{44\mathrm{IN}}$, $\lambda_{1\mathrm{IN}}$ and $\xi_{1\mathrm{IN}}$, $\xi_{3\mathrm{IN}}$, $\xi_{5\mathrm{IN}}$ are given by Equations (1.13); (1.15), (2.16), (5.4) and (5.15), respectively, and $\eta_{1\mathrm{IN}}$, $\kappa_{1\mathrm{IN}}$ (see Equation (5.15)) have the forms

$$\eta_{1IN} = -\frac{p_n \left(\lambda_{1IN} \gamma_{1IN} + \gamma_{2IN} \right)}{\xi_{1IN} \, x_n^{\lambda_{1IN}-1}} - \frac{\gamma_{3IN}}{s_{44IN}} \frac{\partial}{\partial \varphi} \left(\frac{p_n}{\xi_{1IN} \, x_n^{\lambda_{1IN}-1}} \right) - \frac{\gamma_{4IN}}{s_{44IN}} \frac{\partial}{\partial \nu} \left(\frac{p_n}{\xi_{1IN} \, x_n^{\lambda_{1IN}-1}} \right),$$

$$\kappa_{1IN} = \xi_{5IN} \left(\frac{p_n}{\xi_{1IN} \, x_{IN}^{\lambda_{1IN}-1}} \right)^2 + \frac{1}{s_{44M}} \left[\frac{\partial}{\partial \varphi} \left(\frac{p_n}{\xi_{1IN} \, x_{IN}^{\lambda_{1IN}-1}} \right) \right]^2 + \frac{\Theta^2}{s_{44M}} \left[\frac{\partial}{\partial \nu} \left(\frac{p_n}{\xi_{1IN} \, x_{IN}^{\lambda_{1IN}-1}} \right) \right]^2 . \tag{5.36}$$

The normal stress p_{n} is given by Equation (4.6). With regard to Equation (5.28), the coefficient ρ_{IN} in Equation (4.6) is derived as

$$\rho_{IN} = \frac{\left(1+\mu_{IN}\right)\left(1-2\mu_{IN}\right)}{E_{IN}\left[\lambda_{1IN}\left(1-\mu_{IN}\right)+2\mu_{IN}\right]} . \tag{5.37}$$

CHAPTER 6

MATHEMATICAL MODEL 2

6.1 Mathematical procedure 2

If the mathematical procedure ∂^2Eq.(2.27)/∂x_n^2 is performed, then we get

$$x_n \frac{\partial^3 U_n}{\partial x_n^3} + (2 - c_3) \frac{\partial^2 U_n}{\partial x_n^2} = 0 , \tag{6.1}$$

where $c_3 < 0$, $U_n = U_n(x_n, \varphi, \theta)$ are given by Equation (2.21), (2.28), respectively. If $U_n = x_n^{\lambda}$, then, with respect to Equation (6.1), we get

$$U_n = C_1 x_n + C_2 x_n^{c3} + C_3 , \tag{6.2}$$

where the integration constants C_1, C_2, C_3 are determined by the boundary conditions in Chapter 4. If Equation (6.2) is substituted to Equation (2.26), we get

$$x_n^2 \frac{\partial^2 u_n}{\partial x_n^2} + 2x_n \frac{\partial u_n}{\partial x_n} - 2u_n = C_1 x_n + C_2 x_n^{c3} + C_3 , \tag{6.3}$$

Using Wronskian's method in Section 12.1 (Rektorys, 1973, 225-227), the mathemati-cal solution of Equation (6.3) has the form

$$u_n = C_1 x_n \left(\frac{1}{3} - \ln x_n \right) + C_2 x_n^{c_3} + C_3 . \tag{6.4}$$

With regard to Equation (2.1)-(2.4), (2.17)-(2.20), (2.29), (6.4), we get

$$\varepsilon_n = -C_1 x_n \left(\frac{2}{3} + \ln x_n \right) + C_2 c_3 x_n^{c_3 - 1} , \tag{6.5}$$

$$\varepsilon_\varphi = \varepsilon_\nu = C_1\, x_n \left(\frac{1}{3} - \ln x_n\right) + C_2\, x_n^{c_3 - 1} + \frac{C_3}{x_n}\,, \tag{6.6}$$

$$\varepsilon_{n\varphi} = s_{44}\, \sigma_{n\varphi} = \left(\frac{1}{3} - \ln x_n\right)\frac{\partial C_1}{\partial \varphi} + x_n^{c_3 - 1}\,\frac{\partial C_2}{\partial \varphi} + \frac{1}{x_n}\,\frac{\partial C_3}{\partial \varphi}\,, \tag{6.7}$$

$$\varepsilon_{n\theta} = s_{44}\, \sigma_{n\theta} = \Theta \left[\left(\frac{1}{3} - \ln x_n\right)\frac{\partial C_1}{\partial \nu} + x_n^{c_3 - 1}\,\frac{\partial C_2}{\partial \nu} + \frac{1}{x_n}\,\frac{\partial C_3}{\partial \nu}\right], \tag{6.8}$$

$$\sigma_n = -C_1 \left[\frac{2(c_1 + 2c_2)}{3} + (c_1 - c_2)\ln x_n\right],$$

$$+ C_2 \left[(c_1 + c_2)c_3 - 2c_2\right] x_n^{c_3 - 1} - \frac{2C_3\, c_2}{x_n}\,, \tag{6.9}$$

$$\sigma_\varphi = \sigma_\nu = C_1 \left[\frac{c_1 + 2c_2}{3} - (c_1 - c_2)\ln x_n\right] + C_2\,(c_1 - c_2 c_3)\, x_n^{c_3 - 1} + \frac{C_3 c_1}{x_n}\,, \tag{6.10}$$

$$\sigma_1 = \eta_1 + \eta_2\, \ln x_n + \eta_3\, x_n^{c_3 - 1} + \frac{\eta_4}{x_n}\,, \tag{6.11}$$

$$w = C_1^2\, \kappa_1 + C_2^2\, \kappa_2 + C_3^2\, \kappa_3 + C_1\, C_2\, \kappa_4 + C_1\, C_3\, \kappa_5 + C_2\, C_3\, \kappa_6$$

$$+ \frac{\chi_1}{s_{44}}\left[\left(\frac{\partial C_1}{\partial \varphi}\right)^2 + \Theta^2 \left(\frac{\partial C_1}{\partial \nu}\right)^2\right] + \frac{\chi_2}{s_{44}}\left[\left(\frac{\partial C_2}{\partial \varphi}\right)^2 + \Theta^2 \left(\frac{\partial C_2}{\partial \nu}\right)^2\right]$$

$$+ \frac{\chi_3}{s_{44}}\left[\left(\frac{\partial C_3}{\partial \varphi}\right)^2 + \Theta^2 \left(\frac{\partial C_3}{\partial \nu}\right)^2\right] + \frac{\chi_4}{s_{44}}\left(\frac{\partial C_1}{\partial \varphi}\frac{\partial C_2}{\partial \varphi} + \Theta^2\, \frac{\partial C_1}{\partial \nu}\frac{\partial C_2}{\partial \nu}\right)$$

$$+ \frac{\chi_5}{s_{44}}\left(\frac{\partial C_1}{\partial \varphi}\frac{\partial C_3}{\partial \varphi} + \Theta^2\, \frac{\partial C_1}{\partial \nu}\frac{\partial C_3}{\partial \nu}\right) + \frac{\chi_6}{s_{44}}\left(\frac{\partial C_2}{\partial \varphi}\frac{\partial C_3}{\partial \varphi} + \Theta^2\, \frac{\partial C_2}{\partial \nu}\frac{\partial C_3}{\partial \nu}\right), \tag{6.12}$$

where Θ; s_{44}, c_i (i = 1,2,3) are given by Equations (1.13); (2.16), (2.21), respectively, and η_j, κ_k, η_k (j = 1,…,4; k = 1,…,6) are derived as

$$\eta_1 = \frac{1}{3}\left[C_1(\gamma_1 - 2\gamma_2) + \frac{1}{s_{44}}\left(\gamma_3\, \frac{\partial C_1}{\partial \varphi} + \gamma_4\, \frac{\partial C_1}{\partial \nu}\right)\right],$$

$$\eta_2 = -\left[C_1(\gamma_1+\gamma_2) + \frac{1}{s_{44}}\left(\gamma_3 \frac{\partial C_1}{\partial \varphi} + \gamma_4 \frac{\partial C_1}{\partial \nu} \right) \right],$$

$$\eta_3 = C_2(\gamma_1 c_3+\gamma_2) + \frac{1}{s_{44}}\left(\gamma_3 \frac{\partial C_2}{\partial \varphi} + \gamma_4 \frac{\partial C_2}{\partial \nu} \right),$$

$$\eta_4 = C_3 \gamma_2 + \frac{1}{s_{44}}\left(\gamma_3 \frac{\partial C_3}{\partial \varphi} + \gamma_4 \frac{\partial C_3}{\partial \nu} \right),$$

$$\kappa_1 = (c_2 - c_1)\left(\frac{1}{2}\ln x_n + \frac{2}{3} \right) \ln x_n + \frac{7c_1+2c_2}{9} \;,$$

$$\kappa_2 = \left[\frac{c_3^2(c_1+c_2)}{2} + c_1(1-2c_3) \right] x_n^{2(c_3-1)}, \quad \kappa_3 = \frac{c_1}{x_n^2},$$

$$\kappa_4 = \left\{ c_3(c_1-c_2)\ln x_n + 2\left[c_1 - \frac{c_3(2c_1+c_2)}{3} \right] \right\} x_n^{c_3-1},$$

$$\kappa_5 = \frac{2c_1}{x_n}, \quad \kappa_6 = 0 \,,$$

$$\chi_1 = \ln^2 x_n - \frac{2}{3}\ln x_n + \frac{1}{9}, \quad \chi_2 = x_n^{2(c_3-1)}, \quad \chi_3 = \frac{1}{x_n^2},$$

$$\chi_4 = 2\left(\frac{1}{3} - \ln x_n \right) x_n^{c_3-1}, \quad \chi_5 = \frac{2}{x_n}\left(\frac{1}{3} - \ln x_n \right), \quad \chi_6 = x_n^{c_3-2}. \qquad (6.13)$$

The integrals Φ_i, Ψ_i of $\kappa_i = \kappa_i(x_n)$, $\chi_i = \chi_i(x_n)$ ($i = 1,\ldots,6$), respectively, are derived as

$$\Phi_i = \int_{x_{IN}}^{x_M} \kappa_i \, x_n^2 \, dx_n, \quad \Psi_i = \int_{x_{IN}}^{x_M} \chi_i \, x_n^2 \, dx_n, \quad i = 1,\ldots,6 \,, \qquad (6.14)$$

where x_{IN}, x_M are given by Equation (1.15). The following integrals are determined by the equations in Section 12.3, and then we get

$$\Phi_1 = \frac{c_1-c_2}{6}\left\{ x_M^3\left[\left(\ln x_M - \frac{1}{3} \right) + \frac{1}{9} \right] - x_{IN}^3\left[\left(\ln x_{IN} - \frac{1}{3} \right) + \frac{1}{9} \right] \right\}$$

$$+\frac{2(c_1-c_2)}{9}\left\{x_M^3\left(\ln x_M-\frac{1}{3}\right)-x_{IN}^3\left(\ln x_{IN}-\frac{1}{3}\right)\right\}$$
$$+\frac{(7c_1+2c_2)\left(x_M^3-x_{IN}^3\right)}{27},$$
$$\Phi_2=\frac{1}{2c_3+1}\left[\frac{c_3^2(c_1+c_2)}{2}+c_1(1-2c_3)\right]\left(x_M^{2c_3+1}-x_{IN}^{2c_3+1}\right),$$
$$\Phi_3=c_1(x_M-x_{IN}),$$
$$\Phi_4=\frac{c_3(c_1-c_2)}{c_3+2}\left[x_M^{c_3+2}\left(\ln x_M-\frac{1}{c_3+2}\right)-x_{IN}^{c_3+2}\left(\ln x_{IN}-\frac{1}{c_3+2}\right)\right]$$
$$+\frac{2}{c_3+2}\left[c_1-\frac{c_3(2c_1+c_2)}{3}\right]\left(x_M^{c_3+2}-x_{IN}^{c_3+2}\right),$$
$$\Phi_5=c_1\left(x_M^2-x_{IN}^2\right),\quad \Phi_6=0,$$
$$\Psi_1=\frac{x_M^3}{3}\left[(\ln x_M-1)\left(\ln x_M-\frac{1}{3}\right)+\frac{2}{9}\right]$$
$$-\frac{x_{IN}^3}{3}\left[(\ln x_{IN}-1)\left(\ln x_{IN}-\frac{1}{3}\right)+\frac{2}{9}\right],$$
$$\Psi_2=\frac{x_M^{2c_3+1}-x_{IN}^{2c_3+1}}{2c_3+1},\quad \Psi_3=x_M-x_{IN},$$
$$\Psi_4=\frac{2}{c_3+2}\left\{x_M^{c_3+2}\left[\frac{c_3+5}{3(c_3+2)}-\ln x_M\right]-x_{IN}^{c_3+2}\left[\frac{c_3+5}{3(c_3+2)}-\ln x_{IN}\right]\right\},$$
$$\Psi_5=x_M^2\left(\frac{5}{6}-\ln x_M\right)-x_{IN}^2\left(\frac{5}{6}-\ln x_{IN}\right),\quad \Psi_6=\frac{x_M^{c_3+1}-x_{IN}^{c_3+1}}{c_3+1}. \quad (6.15)$$

In case of the ellipsoidal inclusion, we get $(\varepsilon_{nIN})_{x_n=0}=\pm\infty$, $(\sigma_{nIN})_{x_n=0}=\pm\infty$ due to $(\ln x_n)_{x_n=0}=-\infty$, $\left(x_n^{c_{3IN}}\right)_{x_n=0}=\infty$, and then the mathematical solutions (6.4)-(6.12) are suitable for the matrix.

6.2 Cell Matrix

The integration constants C_{1M}, C_{2M}, C_{3M} are determined by Equations (4.1), (4.2) or (4.1)-(4.3), and then the following combina-tions are considered: $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} = 0$; $C_{1M} \neq 0$, $C_{3M} \neq 0$, $C_{2M} = 0$; $C_{2M} \neq 0$, $C_{3M} \neq 0$, $C_{1M} = 0$; $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} \neq 0$. Finally, such a combination is considered to exhibit mininum potential energy $W_p = W_{IN} + W_M$ (see Equation (2.30)).

Conditions $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} = 0$. With regard to Equations (2.30), (4.1), (4.2), (6.4)-(6.12), we get

$$\varepsilon_{nM} = \frac{p_n}{\zeta}\left[\frac{2}{3} + \ln x_n + c_{3M}\left(\frac{1}{3} - \ln x_n\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right], \tag{6.16}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = \frac{u_{nM}}{x_n} =$$

$$= -\frac{p_n}{\zeta}\left[\frac{1}{3} - \ln x_n - \left(\frac{1}{3} - \ln x_n\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right], \tag{6.17}$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = \left(\ln x_n - \frac{1}{3}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right),$$

$$+ x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left[\left(\frac{1}{3} - \ln x_n\right)\left(\frac{x_n}{x_M}\right)\right], \tag{6.18}$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = \Theta\left\{\left(\ln x_n - \frac{1}{3}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right),\right.$$

$$\left. + x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left[\left(\frac{1}{3} - \ln x_n\right)\left(\frac{x_n}{x_M}\right)\right]\right\}, \tag{6.19}$$

$$\sigma_{nM} = \frac{p_n}{\zeta}\left\{\frac{2(c_{1M} + 2c_{2M})}{3} + (c_{1M} - c_{2M})\ln x_n\right.$$

$$\left. + \left[(c_{1M} + c_{2M})c_{3M} - 2c_{2M}\right]\left(\frac{1}{3} - \ln x_M\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right\}, \tag{6.20}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta}\left[\frac{c_{1M}+2c_{2M}}{3} - \left(c_{1M}-c_{2M}\right)\ln x_n + \left(c_{1M}-c_{2M}c_{3M}\right)\left(\frac{1}{3}-\ln x_M\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right], \quad (6.21)$$

$$\sigma_{1M} = \eta_{1M} + \eta_{2M}\ \ln x_n + \eta_{3M}\ x_n^{c_{3M}-1}, \quad (6.22)$$

$$w_M = \left(\frac{p_n}{\zeta}\right)^2\left[\kappa_{1M} + \kappa_{2M}\left(\frac{1-3\ln x_M}{3x_M^{c_{3M}-1}}\right)^2 - \kappa_{4M}\left(\frac{1-3\ln x_M}{3x_M^{c_{3M}-1}}\right)\right]$$
$$+\frac{\chi_{1M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right]^2\right\}$$
$$+\frac{\chi_{2M}}{s_{44M}}\left\{\left\{\frac{\partial}{\partial\varphi}\left[p_n\left(\frac{1-3\ln x_M}{3x_M^{c_{3M}-1}}\right)\right]\right\}^2 + \Theta^2\left\{\frac{\partial}{\partial\nu}\left[p_n\left(\frac{1-3\ln x_M}{3x_M^{c_{3M}-1}}\right)\right]\right\}^2\right\}$$
$$+\frac{\chi_{4M}}{s_{44M}}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left[p_n\left(\frac{1-3\ln x_M}{3x_M^{c_{3M}-1}}\right)\right]$$
$$+\frac{\chi_{4M}\,\Theta^2}{s_{44M}}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\nu}\left[p_n\left(\frac{1-3\ln x_M}{3x_M^{c_{3M}-1}}\right)\right], \quad (6.23)$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{p_n}{\zeta}\right)^2\left[\Phi_{1M} + \Phi_{2M}\left(\frac{1-3\ln x_M}{3x_M^{c_{3M}-1}}\right)^2 - \Phi_{4M}\left(\frac{1-3\ln x_M}{3x_M^{c_{3M}-1}}\right)\right]\Omega\, d\varphi d\nu$$
$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{1M}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right]^2\right\}\Omega d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{2M}\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta x_M^{c_{3M}-1}}\left(\frac{1}{3}-\ln x_M\right)\right]\right\}^2\Omega\,d\varphi dv$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{2M}\,\Theta^2\left\{\frac{\partial}{\partial v}\left[\frac{p_n}{\zeta x_M^{c_{3M}-1}}\left(\frac{1}{3}-\ln x_M\right)\right]\right\}^2\Omega\,d\varphi dv$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{4M}\,\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n(3\ln x_M-1)}{3\zeta\,x_M^{c_{3M}-1}}\right]\Omega d\varphi dv$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{4M}\,\Theta^2\,\frac{\partial}{\partial v}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial v}\left[\frac{p_n(3\ln x_M-1)}{3\zeta\,x_M^{c_{3M}-1}}\right]\Omega\,d\varphi dv$$

(6.24)

where Θ, Ω; x_M; s_{44M}, c_{iM} (i = 1,2,3) and κ_{jM}, χ_{jM}; Φ_{jM}, Ψ_{jM} (j = 1,2,4) are given by Equations (1.13), (1.15), (2.16), (2.21) and (6.13); (6.15), respectively, and ς, ς_i (i = 1,2), η_{jM} (j = 1,2,3; see Equation (6.13)) have the forms

$$\zeta=\zeta_2-\zeta_1\left(\frac{1}{3}-\ln x_M\right),\quad \zeta_1=\left[(c_{1M}+c_{2M})c_{3M}-2c_{2M}\right]\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1},$$

$$\zeta_2=\left[\frac{2(c_{1M}+2c_{2M})}{3}+(c_{1M}-c_{2M})\ln x_{IN}\right],$$

$$\eta_{1M}=-\frac{1}{3}\left[\frac{p_n(\gamma_{1M}-2\gamma_{2M})}{\zeta}+\frac{1}{s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi}+\gamma_{4M}\frac{\partial}{\partial v}\right)\left(\frac{p_n}{\zeta}\right)\right],$$

$$\eta_{2M}=\frac{p_n(\gamma_{1M}+\gamma_{2M})}{\zeta}+\frac{1}{s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi}+\gamma_{4M}\frac{\partial}{\partial v}\right)\left(\frac{p_n}{\zeta}\right),$$

$$\eta_{3M}=\frac{p_n(\gamma_{1M}\,c_{3M}+\gamma_{2M})}{\zeta\,x_M^{c_{3M}-1}}\left(\frac{1}{3}-\ln x_M\right)$$

$$+\frac{1}{s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi}+\gamma_{4M}\frac{\partial}{\partial v}\right)\left[\frac{p_n}{\zeta\,x_M^{c_{3M}-1}}\left(\frac{1}{3}-\ln x_M\right)\right]. \qquad (6.25)$$

The normal stress p_n is given by Equation (4.6). With regard to Equa-tion (6.17), the coefficient ρ_M in Equation (4.6) is derived as

$$\rho_M = \frac{1}{\zeta}\left[\frac{1}{3} - \ln x_{IN} - \left(\frac{1}{3} - \ln x_{IN}\right)\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1}\right]. \tag{6.26}$$

Conditions $C_{1M} \neq 0$, $C_{3M} \neq 0$, $C_{2M} = 0$. With regard to Equations (2.30), (4.1), (4.2), (6.4)-(6.12), we get

$$\varepsilon_{nM} = \frac{p_n}{\zeta x_M}\left(\frac{2}{3} + \ln x_n\right), \tag{6.27}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = \frac{u_{nM}}{x_n} = $$

$$= -\frac{p_n}{\zeta}\left[\frac{1}{x_M}\left(\frac{1}{3} - \ln x_n\right) - \frac{1}{x_n}\left(\frac{1}{3} - \ln x_M\right)\right], \tag{6.28}$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = \left(\ln x_n - \frac{1}{3}\right)\frac{\partial}{\partial \varphi}\left(\frac{p_n}{\zeta\, x_M}\right)$$

$$+\frac{1}{x_n}\frac{\partial}{\partial \varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{3} - \ln x_M\right)\right] \tag{6.29}$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = \Theta\left\{\left(\ln x_n - \frac{1}{3}\right)\frac{\partial}{\partial \nu}\left(\frac{p_n}{\zeta\, x_M}\right),\right.$$

$$\left.+\frac{1}{x_n}\frac{\partial}{\partial \nu}\left[\frac{p_n}{\zeta}\left(\frac{1}{3} - \ln x_M\right)\right]\right\}, \tag{6.30}$$

$$\sigma_{nM} = \frac{p_n}{\zeta}\left\{\frac{1}{x_M}\left[\frac{2\left(c_{1M} + 2c_{2M}\right)}{3} + \left(c_{1M} - c_{2M}\right)\ln x_n\right]\right.$$

$$\left.-\frac{2c_{2M}}{x_n}\left(\frac{1}{3} - \ln x_M\right)\right\}, \tag{6.31}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta}\left\{\left[\frac{c_{1M} + 2c_{2M}}{3} - (c_{1M} - c_{2M})\ln x_n\right] - \frac{c_{1M}}{x_n}\left(\frac{1}{3} - \ln x_M\right)\right\}, \tag{6.32}$$

$$\sigma_{1M} = \eta_{1M} + \eta_{2M}\ \ln x_n + \frac{\eta_{4M}}{x_n}, \tag{6.33}$$

$$\begin{aligned} w_M = {} & \left(\frac{p_n}{\zeta}\right)^2\left[\frac{\kappa_{1M}}{x_M^2} + \kappa_{3M}\left(\frac{1}{3} - \ln x_M\right)^2 - \frac{\kappa_{5M}}{x_M}\left(\frac{1}{3} - \ln x_M\right)\right] \\ & + \frac{\chi_{1M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M}\right)\right]^2\right\} \\ & + \frac{\chi_{2M}}{s_{44M}}\left\{\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{3} - \ln x_M\right)\right]\right\}^2 + \Theta^2\left\{\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta}\left(\frac{1}{3} - \ln x_M\right)\right]\right\}^2\right\} \\ & + \frac{\chi_{5M}}{s_{44M}}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M}\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\ln x_M - \frac{1}{3}\right)\right] \\ & + \frac{\chi_{5M}\,\Theta^2}{s_{44M}}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M}\right)\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta}\left(\ln x_M - \frac{1}{3}\right)\right], \end{aligned} \tag{6.34}$$

$$\begin{aligned} W_M = {} & 4\int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{p_n}{\zeta}\right)^2\left[\frac{\Phi_{1M}}{x_M^2} + \Phi_{3M}\left(\frac{1}{3} - \ln x_M\right)^2 + \frac{\Phi_{5M}}{x_M}\left(\ln x_M - \frac{1}{3}\right)\right]\Omega\, d\varphi\, d\nu \\ & + \frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{1M}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M}\right)\right]^2\right\}\Omega\, d\varphi\, d\nu \\ & + \frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{3M}\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{3} - \ln x_M\right)\right]\right\}^2\Omega\, d\varphi\, d\nu \end{aligned}$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{3M}\Theta^2\left\{\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta}\left(\frac{1}{3}-\ln x_M\right)\right]\right\}^2\Omega\, d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{5M}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M}\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n(3\ln x_M-1)}{3\zeta}\right]\Omega\, d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{5M}\,\Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M}\right)\frac{\partial}{\partial\nu}\left[\frac{p_n(3\ln x_M-1)}{3\zeta}\right]\Omega\; d\varphi d\nu\,, \tag{6.35}$$

where Θ, Ω; x_M; s_{44M}, c_{iM} ($i = 1,2,3$) and κ_{jM}, χ_{jM}; Φ_{jM}, Ψ_{jM} ($j = 1,2,5$) are given by Equations (1.13), (1.15), (2.16), (2.21) and (6.13); (6.15), respectively, and ς, η_{iM} ($i = 1,2,4$; see Equation (6.13)) have the forms

$$\zeta=\frac{1}{x_M}\left[\frac{2(c_{1M}+2c_{2M})}{3}+(c_{1M}-c_{2M})\ln x_{IN}\right]+\frac{2c_{2M}}{x_{IN}}\left(\frac{1}{3}-\ln x_M\right),$$

$$\eta_{1M}=-\frac{1}{3}\left[\frac{p_n(\gamma_{1M}-2\gamma_{2M})}{\zeta\, x_M}+\frac{1}{s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi}+\gamma_{4M}\frac{\partial}{\partial\nu}\right)\left(\frac{p_n}{\zeta\, x_M}\right)\right],$$

$$\eta_{2M}=\frac{p_n(\gamma_{1M}+\gamma_{2M})}{\zeta\, x_M}+\frac{1}{s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi}+\gamma_{4M}\frac{\partial}{\partial\nu}\right)\left(\frac{p_n}{\zeta\, x_M}\right),$$

$$\eta_{4M}=\frac{p_n\gamma_{2M}}{\zeta}\left(\frac{1}{3}-\ln x_M\right)$$

$$+\frac{1}{s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi}+\gamma_{4M}\frac{\partial}{\partial\nu}\right)\left[\frac{p_n}{\zeta}\left(\frac{1}{3}-\ln x_M\right)\right]. \tag{6.36}$$

The normal stress p_n is given by Equation (4.6). With regard to Equa-tion (6.28), the coefficient ρ_M in Equation (4.6) is derived as

$$\rho_M=\frac{1}{\zeta}\left[\frac{1}{x_M}\left(\frac{1}{3}-\ln x_{IN}\right)-\frac{1}{x_{IN}}\left(\frac{1}{3}-\ln x_M\right)\right]. \tag{6.37}$$

Conditions $C_{2M} \neq 0$, $C_{3M} \neq 0$, $C_{1M} = 0$. With regard to Equations (2.30), (4.1), (4.2), (6.4)-(6.12), we get

$$\varepsilon_{nM} = \frac{p_n\, c_{3M}}{\zeta\, x_M}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}, \tag{6.38}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = \frac{u_{nM}}{x_n} = -\frac{p_n}{\zeta\, x_n}\left[1-\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right], \tag{6.39}$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = -\frac{1}{x_n}\left[x_n^{c_{3M}}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}}}\right)-\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right], \tag{6.40}$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = -\frac{\Theta}{x_n}\left[x_n^{c_{3M}}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}}}\right)-\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right], \tag{6.41}$$

$$\sigma_{nM} = -\frac{p_n}{\zeta}\left[\frac{c_{3M}\left(c_{1M}+c_{2M}\right)-2c_{2M}}{x_M}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}+\frac{2c_{2M}}{x_n}\right], \tag{6.42}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta}\left[\frac{c_{1M}+c_{2M}c_{3M}}{x_M}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}-\frac{c_{1M}}{x_n}\right], \tag{6.43}$$

$$\sigma_{1M} = \eta_{3M}\, x_n^{c_{3M}-1}+\frac{\eta_{4M}}{x_n}, \tag{6.44}$$

$$\begin{aligned} w_M = &\left(\frac{p_n}{\zeta}\right)^2\left(\frac{\kappa_{2M}}{x_M^{2c_{3M}}}+\kappa_{3M}-\frac{\kappa_{6M}}{x_M^{c_{3M}}}\right) \\ &+\frac{\chi_{2M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}}}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}}}\right)\right]^2\right\} \\ &+\frac{\chi_{3M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right]^2\right\} \\ &+\frac{\chi_{6M}}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}}}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)+\Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}}}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right], \end{aligned} \tag{6.45}$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{p_n}{\zeta}\right)^2\left(\frac{\Psi_{2M}}{x_M^{2c_{3M}}}+\Psi_{3M}-\frac{\Psi_{6M}}{x_M^{c_{3M}}}\right)\Omega d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{2M}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}}}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}}}\right)\right]^2\right\}\Omega d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{3M}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right]^2\right\}\Omega d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{2M}\,\Theta^2\left\{\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta x_M^{c_{3M}-1}}\left(\frac{1}{3}-\ln x_M\right)\right]\right\}^2\Omega\, d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{6M}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}}}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right]\Omega d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{6M}\,\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}}}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right]\Omega d\varphi d\nu\,, \tag{6.46}$$

where Θ, Ω; x_M; s_{44M}, c_{iM} (i = 1,2,3) and κ_{jM}, χ_{jM}; Φ_{jM}, Ψ_{jM} (j = 2,3,6) are given by Equations (1.13), (1.15), (2.16), (2.21) and (6.13); (6.15), respectively, and ς, η_{iM} (i = 2,3; see Equation (6.13)) have the forms

$$\zeta = \frac{1}{x_{IN}}\left\{\left[\left(c_{1M}+c_{2M}\right)c_{3M}-2c_{2M}\right]\left(\frac{x_{IN}}{x_{\mathrm{Im}}}\right)^{c_{3M}-1}+2c_{2M}\right\},$$

$$\eta_{3M} = -\frac{p_n\left(\gamma_{1M}\,c_{3M}+\gamma_{2M}\right)}{\zeta\, x_M^{c_{3M}}}-\frac{1}{s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi}-\gamma_{4M}\frac{\partial}{\partial\nu}\right)\left(\frac{p_n}{\zeta\, x_M^{c_{3M}}}\right),$$

$$\eta_{4M} = \frac{p_n \gamma_{2M}}{\zeta} + \frac{1}{s_{44M}} \left(\gamma_{3M} \frac{\partial}{\partial \varphi} + \gamma_{4M} \frac{\partial}{\partial \nu} \right) \left(\frac{p_n}{\zeta} \right). \tag{6.47}$$

The normal stress p_n is given by Equation (4.6). With regard to Equa-tion (6.39), the coefficient ρ_M in Equation (4.6) is derived as

$$\rho_M = \frac{1}{\zeta} \left[\left(\frac{x_{IN}}{x_M} \right)^{c_{3M}} - 1 \right]. \tag{6.48}$$

Conditions $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} \neq 0$. With regard to Equations (2.30), (4.1)-(4.3), (6.4)-(6.12), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta} \left[\zeta_1 \left(\frac{2}{3} + \ln x_n \right) - \zeta_2 \, c_{3M} x_n^{c_{3M}-1} \right], \tag{6.49}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = \frac{u_{nM}}{x_n} = \frac{p_n}{\zeta} \left[\zeta_1 \left(\frac{1}{3} - \ln x_n \right) + \zeta_2 \, x_n^{c_{3M}-1} + \frac{\zeta_3}{x_n} \right], \tag{6.50}$$

$$\varepsilon_{n\varphi M} = s_{44M} \, \sigma_{n\varphi M} = \left(\frac{1}{3} - \ln x_n \right) \frac{\partial}{\partial \varphi} \left(\frac{p_n \zeta_1}{\zeta} \right) + x_n^{c_{3M}-1} \frac{\partial}{\partial \varphi} \left(\frac{p_n \zeta_2}{\zeta} \right)$$
$$+ \frac{1}{x_n} \frac{\partial}{\partial \varphi} \left(\frac{p_n \zeta_3}{\zeta} \right), \tag{6.51}$$

$$\varepsilon_{n\theta M} = s_{44M} \, \sigma_{n\theta M} = \Theta \left[\left(\frac{1}{3} - \ln x_n \right) \frac{\partial}{\partial \varphi} \left(\frac{p_n \zeta_1}{\zeta} \right) + x_n^{c_{3M}-1} \frac{\partial}{\partial \varphi} \left(\frac{p_n \zeta_2}{\zeta} \right), \right.$$
$$\left. + \frac{1}{x_n} \frac{\partial}{\partial \varphi} \left(\frac{p_n \zeta_3}{\zeta} \right) \right], \tag{6.52}$$

$$\sigma_{nM} = -\frac{p_n}{\zeta} \left\{ \zeta_1 \left[\frac{2(c_{1M} + 2c_{2M})}{3} + (c_{1M} - c_{2M}) \ln x_n \right] \right.$$
$$\left. - \zeta_2 \left[(c_{1M} + c_{2M}) c_{3M} - 2c_{2M} \right] x_n^{c_{3M}-1} + \frac{2 c_{2M} \zeta_3}{x_n} \right\} \tag{6.53}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta}\left\{\zeta_1\left[\frac{c_{1M} + 2c_{2M}}{3} - \left(c_{1M} - c_{2M}\right)\ln x_n\right]\right.$$

$$\left. + \zeta_2\left(c_{1M} - c_{2M}c_{3M}\right)x_n^{c_{3M}-1} + \frac{c_{1M}\,\zeta_3}{x_n}\right\}, \quad (6.54)$$

$$\sigma_{1M} = \eta_{1M} + \eta_{2M}\ \ln x_n + \eta_{3M}\ x_n^{c_{3M}-1} + \frac{\eta_{4M}}{x_n}, \quad (6.55)$$

$$w_M = \left(\frac{p_n}{\zeta}\right)^2 (\kappa_{1M}\,\zeta_1^2 + \kappa_{2M}\,\zeta_2^2 + \kappa_{3M}\,\zeta_3^2$$

$$+\kappa_{4M}\zeta_1\zeta_2 + \kappa_{5M}\zeta_1\zeta_3 + \kappa_{6M}\zeta_2\zeta_3)$$

$$+\frac{\chi_{1M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\right]^2\right\}$$

$$+\frac{\chi_{2M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right)\right]^2\right\}$$

$$+\frac{\chi_{3M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]^2\right\}$$

$$+\frac{\chi_{4M}}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right) + \Theta^2\,\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right)\right]$$

$$+\frac{\chi_{5M}}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right) + \Theta^2\,\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]$$

$$+\frac{\chi_{6M}}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right) + \Theta^2\,\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right],$$

$$(6.56)$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{p_n}{\zeta}\right)^2\left(\Phi_{1M}\,\zeta_1^2 + \Phi_{2M}\,\zeta_2^2 + \Phi_{3M}\,\zeta_3^2 + \Phi_{4M}\zeta_1\zeta_2\right.$$

$$\left. + \Phi_{5M}\zeta_1\zeta_3 + \Phi_{6M}\zeta_2\zeta_3\right)\Omega\; d\varphi\; d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{1M}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)\right]^2\right.$$

$$\left.+\,\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\right]^2\right\}\Omega\,d\varphi\,d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{4M}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right)\right]\Omega\,d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{4M}\,\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right)\right]\Omega\,d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{5M}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]\Omega\,d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{5M}\,\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]\Omega\,d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{6M}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]\Omega\,d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{6M}\,\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]\Omega\,d\varphi d\nu\,,$$

(6.57)

where Θ, Ω; x_M; s_{44M}, c_{iM} (i = 1,2,3) and κ_{jM}, χ_{jM}; Φ_{jM}, Ψ_{jM} (j = 1,…,6) are given by Equations (1.13), (1.15), (2.16), (2.21) and (6.13); (6.15), respectively, and ς, ς_i, η_{iM} (i = 1,2,3; see Equation (6.13)) have the forms

$$\zeta_1 = c_{3M}\,x_M^{c_{3M}-1},\quad \zeta_2 = \frac{2}{3}+\ln x_M\,,$$

$$\zeta_3 = -\,x_M^{c_{3M}}\left[\frac{2}{3}+\ln x_M + c_{3M}\left(\frac{1}{3}-\ln x_M\right)\right],$$

$$\zeta = c_{3M}x_M^{c_{3M}-1}\left\{\left[\frac{2(c_{1M}+2c_{2M})}{3}+(c_{1M}-c_{2M})\ln x_{IN}\right]\right.$$

$$
\left. - \frac{2 c_{2M} x_M}{x_{IN}} \left(\frac{1}{3} - \ln x_M \right) \right\}
$$

$$
- \left\{ \left[\left(c_{1M} + c_{2M} \right) c_{3M} - 2 c_{2M} \right] x_{IN}^{c_{3M} - 1} + \frac{2 c_{2M} x_M^{c_{3M}}}{x_{IN}} \right\} \left(\frac{2}{3} + \ln x_M \right),
$$

$$
\eta_{1M} = \frac{1}{3} \left[\frac{p_n \zeta_1 \left(\gamma_{1M} - 2 \gamma_{2M} \right)}{\zeta} + \frac{1}{s_{44M}} \left(\gamma_{3M} \frac{\partial}{\partial \varphi} + \gamma_{4M} \frac{\partial}{\partial \nu} \right) \left(\frac{p_n \zeta_1}{\zeta} \right) \right],
$$

$$
\eta_{2M} = - \left[\frac{p_n \zeta_1 \left(\gamma_{1M} + \gamma_{2M} \right)}{\zeta} + \frac{1}{s_{44M}} \left(\gamma_{3M} \frac{\partial}{\partial \varphi} + \gamma_{4M} \frac{\partial}{\partial \nu} \right) \left(\frac{p_n \zeta_1}{\zeta} \right) \right],
$$

$$
\eta_{3M} = \frac{p_n \zeta_2 \left(\gamma_{1M} c_{3M} + \gamma_{2M} \right)}{\zeta} + \frac{1}{s_{44M}} \left(\gamma_{3M} \frac{\partial}{\partial \varphi} + \gamma_{4M} \frac{\partial}{\partial \nu} \right) \left(\frac{p_n \zeta_2}{\zeta} \right),
$$

$$
\eta_{4M} = \frac{p_n \zeta_3 \gamma_{2M}}{\zeta} + \frac{1}{s_{44M}} \left(\gamma_{3M} \frac{\partial}{\partial \varphi} + \gamma_{4M} \frac{\partial}{\partial \nu} \right) \left(\frac{p_n \zeta_3}{\zeta} \right). \tag{6.58}
$$

The normal stress p_{n} is given by Equation (4.6). With regard to Equa-tion (6.50), the coefficient ρ_{M} in Equation (4.6) is derived as

$$
\rho_M = \frac{1}{\zeta} \left[\zeta_1 \left(\ln x_{IN} - \frac{1}{3} \right) - \zeta_2 x_{IN}^{c_{3M} - 1} - \frac{\zeta_3}{x_{IN}} \right]. \tag{6.59}
$$

CHAPTER 7

MATHEMATICAL MODEL 3

7.1 Mathematical procedure 3

If the mathematical procedure $x_n\ \partial^2$Eq.(2.26)$/\partial x_n^2$ is performed, then we get

$$x_n^3 \frac{\partial^3 u_n}{\partial x_n^3} + 4{x_n}^2 \frac{\partial^2 u_n}{\partial x_n^2} + \frac{x_n}{s_{44}(c_1 + c_2)} \frac{\partial U_n}{\partial x_n} = 0 , \tag{7.1}$$

where s_{44}, c_i (i = 1,2,3), $U_n = U_n\ (x_n,\varphi,\theta)$ are given by Equation (2.16), (2.21), (2.28), respectively. With regard to Equation (2.27), (6.2), we get

$$x_n \frac{\partial U_n}{\partial x_n} = c_3\left(C_1\, x_n + C_2\, x_n^{c_3} + C_3\right), \tag{7.2}$$

where the integration constants C_1, C_2, C_3 are determined by the boundary conditions in Chapter 4. If Equation (7.2) is substituted to Equation (7.1), we get

$$x_n^3 \frac{\partial^3 u_n}{\partial x_n^3} + 4{x_n}^2 \frac{\partial^2 u_n}{\partial x_n^2} = C_1\, x_n + C_2\, x_n^{c_3} + C_3 , \tag{7.3}$$

Using Wronskian's method in Section 12.1 (Rektorys, 1973, 225-227), the mathemati-cal solution of Equation (7.3) has the form

$$u_n = C_1\, x_n \left(\frac{4}{3} - \ln x_n\right) + C_2\, x_n^{c_3} + C_3\left(\frac{1}{2} + \ln x_n\right) . \tag{7.4}$$

and with regard to Equation (2.1)-(2.4), (2.17)-(2.20), (2.29), (7.4), we get

$$\varepsilon_n = C_1\left(\frac{1}{3}-\ln x_n\right)+C_2\,c_3\,x_n^{c_3-1}+\frac{C_3}{x_n}\,, \tag{7.5}$$

$$\varepsilon_\varphi = \varepsilon_\nu = C_1\left(\frac{4}{3}-\ln x_n\right)+C_2\,x_n^{c_3-1}+\frac{C_3}{x_n}\left(\frac{1}{2}+\ln x_n\right), \tag{7.6}$$

$$\varepsilon_{n\varphi} = s_{44}\,\sigma_{n\varphi} = \left(\frac{4}{3}-\ln x_n\right)\frac{\partial C_1}{\partial\varphi}+x_n^{c_3-1}\frac{\partial C_2}{\partial\varphi}+\frac{1}{x_n}\left(\frac{1}{2}+\ln x_n\right)\frac{\partial C_3}{\partial\varphi}\,, \tag{7.7}$$

$$\varepsilon_{n\theta} = s_{44}\,\sigma_{n\theta} = \Theta\left[\left(\frac{4}{3}-\ln x_n\right)\frac{\partial C_1}{\partial\varphi}+x_n^{c_3-1}\frac{\partial C_2}{\partial\varphi}+\frac{1}{x_n}\left(\frac{1}{2}+\ln x_n\right)\frac{\partial C_3}{\partial\varphi}\right] \tag{7.8}$$

$$\sigma_n = C_1\left[\frac{c_1-7c_2}{3}-(c_1-c_2)\ln x_n\right]+C_2\left[(c_1+c_2)c_3-2c_2\right]x_n^{c_3-1} + \frac{C_3}{x_n}\left(c_1-2c_2\ln x_n\right), \tag{7.9}$$

$$\sigma_\varphi = \sigma_\nu = C_1\left[\frac{4c_1-c_2}{3}-(c_1-c_2)\ln x_n\right]+C_2\left(c_1-c_2c_3\right)x_n^{c_3-1} + \frac{C_3}{x_n}\left(\frac{c_1-2c_2}{2}-c_1\ln x_n\right), \tag{7.10}$$

$$\sigma_1 = \eta_1+\eta_2\,\ln x_n+\eta_3\,x_n^{c_3-1}+\frac{\eta_4+\eta_5\ln x_n}{x_n}\,, \tag{7.11}$$

$$w = C_1^2\,\kappa_1+C_2^2\,\kappa_2+C_3^2\,\kappa_3+C_1\,C_2\,\kappa_4+C_1\,C_3\,\kappa_5+C_2\,C_3\,\kappa_6 + \frac{\chi_1}{s_{44}}\left[\left(\frac{\partial C_1}{\partial\varphi}\right)^2+\Theta^2\left(\frac{\partial C_1}{\partial\nu}\right)^2\right]+\frac{\chi_2}{s_{44}}\left[\left(\frac{\partial C_2}{\partial\varphi}\right)^2+\Theta^2\left(\frac{\partial C_2}{\partial\nu}\right)^2\right] + \frac{\chi_3}{s_{44}}\left[\left(\frac{\partial C_3}{\partial\varphi}\right)^2+\Theta^2\left(\frac{\partial C_3}{\partial\nu}\right)^2\right]+\frac{\chi_4}{s_{44}}\left(\frac{\partial C_1}{\partial\varphi}\frac{\partial C_2}{\partial\varphi}+\Theta^2\,\frac{\partial C_1}{\partial\nu}\frac{\partial C_2}{\partial\nu}\right)$$

$$+\frac{\chi_5}{s_{44}}\left(\frac{\partial C_1}{\partial \varphi}\frac{\partial C_3}{\partial \varphi}+\Theta^2\frac{\partial C_1}{\partial \nu}\frac{\partial C_3}{\partial \nu}\right)+\frac{\chi_6}{s_{44}}\left(\frac{\partial C_2}{\partial \varphi}\frac{\partial C_3}{\partial \varphi}+\Theta^2\frac{\partial C_2}{\partial \nu}\frac{\partial C_3}{\partial \nu}\right), \tag{7.12}$$

where Θ is given by Equation (1.13); and η_j, κ_k, η_k ($j = 1,\ldots,4$; $k = 1,\ldots,6$) are derived as

$$\eta_1=\frac{1}{3}\left[C_1(\gamma_1+4\gamma_2)+\frac{4}{s_{44}}\left(\gamma_3\frac{\partial C_1}{\partial \varphi}+\gamma_4\frac{\partial C_1}{\partial \nu}\right)\right],$$

$$\eta_2=-\left[C_1(\gamma_1+\gamma_2)+\frac{1}{s_{44}}\left(\gamma_3\frac{\partial C_1}{\partial \varphi}+\gamma_4\frac{\partial C_1}{\partial \nu}\right)\right],$$

$$\eta_3=C_2(\gamma_1 c_3+\gamma_2)+\frac{1}{s_{44}}\left(\gamma_3\frac{\partial C_2}{\partial \varphi}+\gamma_4\frac{\partial C_2}{\partial \nu}\right),$$

$$\eta_4=C_3\left(\gamma_1+\frac{\gamma_2}{2}\right)+\frac{1}{2s_{44}}\left(\gamma_3\frac{\partial C_3}{\partial \varphi}+\gamma_4\frac{\partial C_3}{\partial \nu}\right),$$

$$\eta_5=C_3\gamma_2+\frac{1}{2s_{44}}\left(\gamma_3\frac{\partial C_3}{\partial \varphi}+\gamma_4\frac{\partial C_3}{\partial \nu}\right),$$

$$\kappa_1=\frac{c_2-c_1}{2}\ln^2 x_n+\frac{c_1-c_2}{2}\ln x_n+\frac{17c_1+c_2}{18},$$

$$\kappa_2=\left[\frac{c_3^2(c_1+c_2)}{2}+c_1(1-2c_3)\right]x_n^{2(c_3-1)},$$

$$\kappa_3=\frac{1}{x_n^2}\left[c_1\ln x_n(\ln x_n-1)+\frac{c_2-2c_1}{4}\right],$$

$$\kappa_4=\left\{c_3(c_1-c_2)\ln x_n+2\left[c_1-\frac{c_3(2c_1+c_2)}{3}\right]\right\}x_n^{c_3-1},$$

$$\kappa_5=\frac{1}{x_n}\left[(3c_1-c_2)\ln x_n-\frac{4c_1-c_2}{3}\right],$$

$$\kappa_6=[2c_1(1-c_3)\ln x_n-c_1+c_2c_3]x_n^{c_3-2},$$

$$\chi_1=\ln^2 x_n-\frac{8}{3}\ln x_n+\frac{16}{9},\quad \chi_2=x_n^{2(c_3-1)},$$

$$\chi_3 = \frac{1}{x_n^2}\left[\ln x_n(\ln x_n + 1) + \frac{1}{4}\right], \quad \chi_4 = 2\left(\frac{4}{3} - \ln x_n\right)x_n^{c_3 - 1},$$

$$\chi_5 = \frac{1}{3x_n}\left[4 + \ln x_n(5 - 6\ln x_n)\right], \quad \chi_6 = x_n^{c_3 - 2}(2\ln x_2 + 1). \tag{7.13}$$

With regard to (6.14), (7.13), we get (see Section 12.3)

$$\Phi_1 = \frac{c_1 - c_2}{6}\left\{x_M^3\left[\left(\ln x_M - \frac{1}{3}\right) + \frac{1}{9}\right] - x_{IN}^3\left[\left(\ln x_{IN} - \frac{1}{3}\right) + \frac{1}{9}\right]\right\}$$
$$+ \frac{c_1 - c_2}{9}\left\{x_M^3\left(\ln x_M - \frac{1}{3}\right) - x_{IN}^3\left(\ln x_{IN} - \frac{1}{3}\right)\right\}$$
$$+ \frac{(17c_1 + c_2)\left(x_M^3 - x_{IN}^3\right)}{54},$$

$$\Phi_2 = \frac{1}{2c_3 + 1}\left[\frac{c_3^2(c_1 + c_2)}{2} + c_1(1 - 2c_3)\right]\left(x_M^{2c_3+1} - x_{IN}^{2c_3+1}\right),$$

$$\Phi_3 = c_1\{x_M[\ln x_M(\ln x_M - 2) + 2] - x_M[\ln x_{IN}(\ln x_{IN} - 2) + 2]\}$$
$$- c_1[x_M(\ln x_M - 1) - x_{IN}(\ln x_{IN} - 1)] + \frac{c_2 - 2c_1}{4}(x_M - x_{IN}),$$

$$\Phi_4 = \frac{c_3(c_1 - c_2)}{c_3 + 2}\left[x_M^{c_3+2}\left(\ln x_M - \frac{1}{c_3 + 2}\right) - x_{IN}^{c_3+2}\left(\ln x_{IN} - \frac{1}{c_3 + 2}\right)\right]$$
$$+ \frac{1}{c_3 + 2}\left[2c_1 + \frac{c_3(c_2 - 7c_1)}{3}\right]\left(x_M^{c_3+2} - x_{IN}^{c_3+2}\right),$$

$$\Phi_5 = \frac{3c_1 - c_2}{2}\left[x_M^2\left(\ln x_M - \frac{1}{2}\right) - x_{IN}^2\left(\ln x_{IN} - \frac{1}{2}\right)\right] - \frac{4c_1 - c_2}{6}\left(x_M^2 - x_{IN}^2\right)$$

$$\Phi_6 = \frac{2c_1(1 - c_3)}{c_3 + 1}\left[x_M^{c_3+1}\left(\ln x_M - \frac{1}{c_3 + 1}\right) - x_{IN}^{c_3+1}\left(\ln x_{IN} - \frac{1}{c_3 + 1}\right)\right]$$
$$+ \frac{c_2 c_3 - c_1}{c_3 + 1}\left(x_M^{c_3+1} - x_{IN}^{c_3+1}\right),$$

$$\Psi_1 = \frac{x_M^3}{3}\left[(\ln x_M - 3)\left(\ln x_M - \frac{1}{3}\right) + \frac{17}{9}\right]$$

$$-\frac{x_{IN}^3}{3}\left[(\ln x_{IN}-3)\left(\ln x_{IN}-\frac{1}{3}\right)+\frac{17}{9}\right],\quad \Psi_2=\frac{x_M^{2c_3+1}-x_{IN}^{2c_3+1}}{2c_3+1},$$

$$\Psi_3 = x_M \ln x_M (\ln x_M -1) - x_{IN}\ln x_{IN}(\ln x_{IN}-1)+\frac{5((x_M - x_{IN}))}{4},$$

$$\Psi_4=\frac{2}{c_3+2}\left\{x_M^{c_3+2}\left[\frac{4c_3+11}{3(c_3+2)}-\ln x_M\right]-x_{IN}^{c_3+2}\left[\frac{4c_3+11}{3(c_3+2)}-\ln x_{IN}\right]\right\},$$

$$\Psi_5=\frac{2\left(x_M^2-x_{IN}^2\right)}{3}+\frac{5}{6}\left[x_M^2\left(\ln x_M-\frac{1}{2}\right)-x_{IN}^2\left(\ln x_{IN}-\frac{1}{2}\right)\right]$$

$$-x_M^2\left[\ln x_M(\ln x_M-1)+\frac{1}{2}\right]+x_{IN}^2\left[\ln x_{IN}(\ln x_{IN}-1)+\frac{1}{2}\right],$$

$$\Psi_6=\frac{2}{c_3+1}\left[x_M^{c_3+1}\left(\ln x_M-\frac{1}{c_3+1}\right)-x_{IN}^{c_3+1}\left(\ln x_{IN}-\frac{1}{c_3+1}\right)\right]$$

$$+\frac{1}{c_3+1}\left(x_M^{c_3+1}-x_{IN}^{c_3+1}\right). \tag{7.14}$$

In case of the ellipsoidal inclusion, we get $(\varepsilon_{nIN})_{x_n=0}=\pm\infty$, $(\sigma_{nIN})_{x_n=0}=\pm\infty$ due to $(\ln x_n)_{x_n=0}=-\infty$, $\left(x_n^{c_{3IN}}\right)_{x_n=0}=\infty$, and then the mathematical solutions (7.4)- (7.12) are suitable for the matrix.

7.2 Cell Matrix

The integration constants C_{1M}, C_{2M}, C_{3M} are determined by Equations (4.1), (4.2) or (4.1)-(4.3), and then the following combina-tions are considered: $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} = 0$; $C_{1M} \neq 0$, $C_{3M} \neq 0$, $C_{2M} = 0$; $C_{2M} \neq 0$, $C_{3M} \neq 0$, $C_{1M} = 0$; $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} \neq 0$. Finally, such a combination is considered to exhibit mininum potential energy $W_p = W_{IN} + W_M$ (see Equation (2.30)).

Conditions $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} = 0$. With regard to Equations (2.30), (4.1), (4.2), (7.4)-(7.12), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta}\left[\frac{1}{3} - \ln x_n - c_{3M}\left(\frac{4}{3} - \ln x_n\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right], \tag{7.15}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = \frac{u_{nM}}{x_n}$$
$$= -\frac{p_n}{\zeta}\left[\frac{4}{3} - \ln x_n - \left(\frac{4}{3} - \ln x_n\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right], \tag{7.16}$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = \left(\ln x_n - \frac{4}{3}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)$$
$$+ x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left[\left(\frac{4}{3} - \ln x_n\right)\left(\frac{x_n}{x_M}\right)\right], \tag{7.17}$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = \Theta\left\{\left(\ln x_n - \frac{4}{3}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right.$$
$$\left. + x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left[\left(\frac{4}{3} - \ln x_n\right)\left(\frac{x_n}{x_M}\right)\right]\right\}, \tag{7.18}$$

$$\sigma_{nM} = \frac{p_n}{\zeta}\left\{\frac{7c_{2M} - c_{1M}}{3} + \left(c_{1M} - c_{2M}\right)\ln x_n\right.$$
$$\left. + \left[\left(c_{1M} + c_{2M}\right)c_{3M} - 2c_{2M}\right]\left(\frac{4}{3} - \ln x_M\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right\}, \tag{7.19}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = \frac{p_n}{\zeta}\left[\frac{c_{2M} - 4c_{1M}}{3} + \left(c_{1M} - c_{2M}\right)\ln x_n\right.$$
$$\left. + \left(c_{1M} - c_{2M}c_{3M}\right)\left(\frac{4}{3} - \ln x_M\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right], \tag{7.20}$$

$$\sigma_{1M} = \eta_{1M} + \eta_{2M}\ln x_n + \eta_{3M}\,x_n^{c_{3M}-1}, \tag{7.21}$$

$$w_M = \left(\frac{p_n}{\zeta}\right)^2 \left[\kappa_{1M} + \kappa_{2M}\left(\frac{4-3\ln x_M}{3x_M^{c_{3M}-1}}\right)^2 + \kappa_{4M}\left(\frac{3\ln x_M - 4}{3x_M^{c_{3M}-1}}\right)\right]$$

$$+\frac{\chi_{1M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right]^2\right\}$$

$$+\frac{\chi_{2M}}{s_{44M}}\left\{\left\{\frac{\partial}{\partial\varphi}\left[p_n\left(\frac{4-3\ln x_M}{3x_M^{c_{3M}-1}}\right)\right]\right\}^2 + \Theta^2\left\{\frac{\partial}{\partial\nu}\left[p_n\left(\frac{4-3\ln x_M}{3x_M^{c_{3M}-1}}\right)\right]\right\}^2\right\}$$

$$+\frac{\chi_{4M}}{s_{44M}}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left[p_n\left(\frac{3\ln x_M - 4}{3x_M^{c_{3M}-1}}\right)\right]$$

$$+\frac{\chi_{4M}\,\Theta^2}{s_{44M}}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\nu}\left[p_n\left(\frac{3\ln x_M - 4}{3x_M^{c_{3M}-1}}\right)\right], \qquad (7.22)$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{p_n}{\zeta}\right)^2\left[\Phi_{1M} + \Phi_{2M}\left(\frac{4-3\ln x_M}{3x_M^{c_{3M}-1}}\right)^2\right.$$

$$\left. + \Phi_{4M}\left(\frac{3\ln x_M - 4}{3x_M^{c_{3M}-1}}\right)\right]\Omega\, d\varphi\, d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{1M}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right]^2\right\}\Omega\, d\varphi\, d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{2M}\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta x_M^{c_{3M}-1}}\left(\frac{4}{3} - \ln x_M\right)\right]\right\}^2\Omega\, d\varphi\, d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{2M}\,\Theta^2\left\{\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta x_M^{c_{3M}-1}}\left(\frac{4}{3}-\ln x_M\right)\right]\right\}^2\Omega\,d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{4M}\,\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n(3\ln x_M-4)}{3\zeta\, x_M^{c_{3M}-1}}\right]\Omega\,d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{4M}\,\Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\nu}\left[\frac{p_n(3\ln x_M-4)}{3\zeta\, x_M^{c_{3M}-1}}\right]\Omega\,d\varphi d\nu \qquad (7.23)$$

where Θ, Ω; x_M; s_{44M}, c_{iM} (i = 1,2,3) and κ_{jM}, χ_{jM}; Φ_{jM}, Ψ_{jM} (j = 1,2,4) are given by Equations (1.13), (1.15), (2.16), (2.21) and (7.13); (7.14), respectively, and ς, ς_i (i = 1,2), η_{jM} (j = 1,2,3; see Equation (7.13)) have the forms

$$\zeta=\zeta_2-\zeta_1\left(\frac{4}{3}-\ln x_M\right),\quad \zeta_1=\left[(c_{1M}+c_{2M})c_{3M}-2c_{2M}\right]\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1},$$

$$\zeta_2=\left[\frac{c_{1M}-7c_{2M}}{3}-(c_{1M}-c_{2M})\ln x_{IN}\right],$$

$$\eta_{1M}=-\frac{1}{3}\left[\frac{p_n(\gamma_{1M}+4\gamma_{2M})}{\zeta}+\frac{4}{s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi}+\gamma_{4M}\frac{\partial}{\partial\nu}\right)\left(\frac{p_n}{\zeta}\right)\right],$$

$$\eta_{2M}=\frac{p_n(\gamma_{1M}+\gamma_{2M})}{\zeta}+\frac{1}{s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi}+\gamma_{4M}\frac{\partial}{\partial\nu}\right)\left(\frac{p_n}{\zeta}\right),$$

$$\eta_{3M}=\frac{p_n(\gamma_{1M}c_{3M}+\gamma_{2M})}{\zeta\, x_M^{c_{3M}-1}}\left(\frac{4}{3}-\ln x_M\right)$$

$$+\frac{1}{s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi}+\gamma_{4M}\frac{\partial}{\partial\nu}\right)\left[\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\left(\frac{4}{3}-\ln x_M\right)\right]. \qquad (7.24)$$

The normal stress p_n is given by Equation (4.6). With regard to Equa-tion (7.16), the coefficient ρ_M in Equation (4.6) is derived as

$$\rho_M = \frac{1}{\zeta}\left[\frac{4}{3} - \ln x_{IN} - \left(\frac{4}{3} - \ln x_{IN}\right)\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1}\right]. \tag{7.25}$$

Conditions $C_{1M} \neq 0$, $C_{3M} \neq 0$, $C_{2M} = 0$. With regard to Equations (2.30), (4.1), (4.2), (7.4)-(7.12), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta}\left[\left(\frac{1}{2} + \ln x_M\right)\left(\frac{1}{3} - \ln x_n\right) - \frac{x_M}{x_n}\left(\frac{4}{3} - \ln x_M\right)\right], \tag{7.26}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = \frac{u_{nM}}{x_n}$$

$$= -\frac{p_n}{\zeta}\left[\left(\frac{1}{2} + \ln x_M\right)\left(\frac{4}{3} - \ln x_n\right) - \frac{x_M}{x_n}\left(\frac{4}{3} - \ln x_M\right)\left(\frac{1}{2} + \ln x_n\right)\right], \tag{7.27}$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = \left(\ln x_n - \frac{4}{3}\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2} + \ln x_M\right)\right]$$

$$+\frac{1}{x_n}\left(\frac{1}{2} + \ln x_n\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n x_M}{\zeta}\left(\ln x_M - \frac{4}{3}\right)\right], \tag{7.28}$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = \Theta\left\{\left(\ln x_n - \frac{4}{3}\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2} + \ln x_M\right)\right],\right.$$

$$\left.+\frac{1}{x_n}\left(\frac{1}{2} + \ln x_n\right)\frac{\partial}{\partial\varphi}\left[\frac{p_n x_M}{\zeta}\left(\ln x_M - \frac{4}{3}\right)\right]\right\}, \tag{7.29}$$

$$\sigma_{nM} = \frac{p_n}{\zeta}\left\{\left(\frac{1}{2} + \ln x_M\right)\left[\frac{c_{1M} - 7c_{2M}}{3} - (c_{1M} - c_{2M})\ln x_n\right]\right.$$

$$\left.+\frac{x_M}{x_n}\left(\frac{4}{3} - \ln x_M\right)(c_{1M} - 2c_{2M}\ln x_n)\right\}, \tag{7.30}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = \frac{p_n}{\zeta}\left\{\left(\frac{1}{2} + \ln x_M\right)\left[\frac{4c_{1M} - c_{2M}}{3} - (c_{1M} - c_{2M})\ln x_n\right]\right.$$

$$+\frac{x_M}{x_n}\left(\frac{4}{3}-\ln x_M\right)\left(\frac{c_{1M}-c_{2M}}{2}+c_{3M}\ln x_n\right)\Bigg\}, \quad (7.31)$$

$$\sigma_{1M}=\eta_{1M}+\eta_{2M}\ln x_n+\frac{\eta_{4M}+\eta_{5M}\ln x_n}{x_n}, \quad (7.32)$$

$$w_M=\left(\frac{p_n}{\zeta}\right)^2\left[\kappa_{1M}\left(\frac{1}{2}+\ln x_M\right)+\kappa_{3M}\,x_M^2\left(\frac{4}{3}-\ln x_M\right)^2\right.$$
$$\left.+\kappa_{5M}\,x_M\left(\frac{1}{2}+\ln x_M\right)\left(\frac{4}{3}-\ln x_M\right)\right]$$
$$+\frac{\chi_{1M}}{s_{44M}}\left(\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\right\}^2+\Theta^2\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\right\}^2\right)$$
$$+\frac{\chi_{3M}}{s_{44M}}\left\{\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n\,x_M}{\zeta}\left(\frac{4}{3}-\ln x_M\right)\right]\right\}^2\right.$$
$$\left.+\Theta^2\left\{\frac{\partial}{\partial\nu}\left[\frac{p_n\,x_M}{\zeta}\left(\frac{4}{3}-\ln x_M\right)\right]\right\}^2\right.$$
$$+\frac{\chi_{5M}}{s_{44M}}\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\right\}\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n\,x_M}{\zeta}\left(\frac{4}{3}-\ln x_M\right)\right]\right\}$$
$$+\frac{\chi_{5M}\,\Theta^2}{s_{44M}}\left\{\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\right\}\left\{\frac{\partial}{\partial\nu}\left[\frac{p_n\,x_M}{\zeta}\left(\frac{4}{3}-\ln x_M\right)\right]\right\}, \quad (7.33)$$

$$W_M=4\int\limits_0^{\pi/2}\int\limits_0^{\pi/2}\left(\frac{p_n}{\zeta}\right)^2\left[\left[\Psi_{1M}\left(\frac{1}{2}+\ln x_M\right)+\Psi_{3M}\,x_M^2\left(\frac{4}{3}-\ln x_M\right)^2\right.\right.$$
$$\left.+\Psi_{5M}\,x_M\left(\frac{1}{2}+\ln x_M\right)\left(\frac{4}{3}-\ln x_M\right)\right]\Omega\,d\varphi\,d\nu$$
$$+\frac{4}{s_{44M}}\int\limits_0^{\pi/2}\int\limits_0^{\pi/2}\Psi_{1M}\left(\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\right\}^2\right.$$

$$+\Theta^2\left\{\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\right\}^2\Bigg)\Omega\,d\varphi\,d\nu$$

$$+\frac{4}{s_{44M}}\int\limits_0^{\pi/2}\int\limits_0^{\pi/2}\Psi_{3M}\Bigg(\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n\,x_M}{\zeta}\left(\frac{4}{3}-\ln x_M\right)\right]\right\}^2$$

$$+\Theta^2\left\{\frac{\partial}{\partial\nu}\left[\frac{p_n\,x_M}{\zeta}\left(\frac{4}{3}-\ln x_M\right)\right]\right\}^2\Bigg)\Omega\,d\varphi\,d\nu$$

$$+\frac{4}{s_{44M}}\int\limits_0^{\pi/2}\int\limits_0^{\pi/2}\Psi_{5M}\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\right\}$$

$$\times\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n\,x_M}{\zeta}\left(\frac{4}{3}-\ln x_M\right)\right]\right\}\Omega\,d\varphi\,d\nu$$

$$+\frac{4}{s_{44M}}\int\limits_0^{\pi/2}\int\limits_0^{\pi/2}\Psi_{5M}\,\Theta^2\left\{\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\right\}$$

$$\times\left\{\frac{\partial}{\partial\nu}\left[\frac{p_n\,x_M}{\zeta}\left(\frac{4}{3}-\ln x_M\right)\right]\right\}\Omega\,d\varphi\,d\nu\,, \tag{7.34}$$

where Θ, Ω; x_M; s_{44M}, c_{iM} (i = 1,2,3) and κ_{jM}, χ_{jM}; Φ_{jM}, Ψ_{jM} (j = 1,3,5) are given by Equations (1.13), (1.15), (2.16), (2.21) and (7.13); (7.14), respectively, and ς, ς_i, η_{jM} (i = 1,2; j = 1,2,4,5; see Equation (7.13)) have the forms

$$\zeta=\frac{\zeta_2}{x_M}\left(\frac{1}{2}+\ln x_M\right)-\zeta_1\left(\frac{4}{3}-\ln x_M\right),\quad \zeta_1=\frac{x_M}{x_{IN}}\left(c_{1M}-2c_{2M}\ln x_{IN}\right),$$

$$\zeta_2=x_M\left[\frac{c_{1M}-7c_{2M}}{3}-\left(c_{1M}-c_{2M}\right)\ln x_{IN}\right],$$

$$\eta_{1M}=-\frac{p_n\left(\gamma_{1M}+4\gamma_{2M}\right)}{3\zeta}\left(\frac{1}{2}+\ln x_M\right)-\frac{4\gamma_{3M}}{3s_{44M}}\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]$$

$$-\frac{4\gamma_{4M}}{3s_{44M}}\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right],$$

$$\eta_{2M}=\frac{p_n(\gamma_{1M}+\gamma_{2M})}{\zeta}\left(\frac{1}{2}+\ln x_M\right)+\frac{\gamma_{3M}}{s_{44M}}\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]$$

$$+\frac{\gamma_{4M}}{s_{44M}}\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right],$$

$$\eta_{4M}=\frac{p_n x_M(2\gamma_{1M}+\gamma_{2M})}{2\zeta}\left(\ln x_M-\frac{4}{3}\right)$$

$$+\frac{\gamma_{3M}}{2s_{44M}}\frac{\partial}{\partial\varphi}\left[\frac{p_n x_M}{\zeta}\left(\ln x_M-\frac{4}{3}\right)\right]$$

$$+\frac{\gamma_{4M}}{2\,s_{44M}}\frac{\partial}{\partial\nu}\left[\frac{p_n x_M}{\zeta}\left(\ln x_M-\frac{4}{3}\right)\right],$$

$$\eta_{5M}=\frac{p_n x_M\gamma_{2M}}{\zeta}\left(\ln x_M-\frac{4}{3}\right)++\frac{\gamma_{3M}}{s_{44M}}\frac{\partial}{\partial\varphi}\left[\frac{p_n x_M}{\zeta}\left(\ln x_M-\frac{4}{3}\right)\right]$$

$$+\frac{\gamma_{4M}}{s_{44M}}\frac{\partial}{\partial\nu}\left[\frac{p_n x_M}{\zeta}\left(\ln x_M-\frac{4}{3}\right)\right]. \tag{7.35}$$

The normal stress p_n is given by Equation (4.6). With regard to Equa-tion (7.27), the coefficient ρ_M in Equation (4.6) is derived as

$$\rho_M=\frac{1}{\zeta}\left[\left(\frac{1}{2}+\ln x_M\right)\left(\frac{4}{3}-\ln x_{IN}\right)-\frac{x_M}{x_{IN}}\left(\frac{1}{2}+\ln x_{IN}\right)\left(\frac{4}{3}-\ln x_M\right)\right]. \tag{7.36}$$

Conditions $C_{2M}\neq 0$, $C_{3M}\neq 0$, $C_{1M}=0$. With regard to Equations (2.30), (4.1), (4.2), (7.4)-(7.12), we get

$$\varepsilon_{nM}=-\frac{p_n}{\zeta}\left[c_{3M}\left(\frac{1}{2}+\ln x_M\right)x_n^{c_{3M}-1}-\frac{x_M^{c_{3M}}}{x_n}\right], \tag{7.37}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = \frac{u_{nM}}{x_n} = -\frac{p_n}{\zeta}\left[\left(\frac{1}{2} + \ln x_M\right)x_n^{c3_M - 1} - \frac{x_M^{c3_M}}{x_n}\left(\frac{1}{2} + \ln x_n\right)\right], \tag{7.38}$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = -x_n^{c3_M - 1}\,\frac{\partial}{\partial \varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2} + \ln x_M\right)\right] + \frac{1}{x_n}\left(\frac{1}{2} + \ln x_n\right)\frac{\partial}{\partial \varphi}\left(\frac{p_n\, x_M^{c3_M}}{\zeta}\right), \tag{7.39}$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = \Theta^2\left\{-x_n^{c3_M - 1}\,\frac{\partial}{\partial \varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2} + \ln x_M\right)\right] + \frac{1}{x_n}\left(\frac{1}{2} + \ln x_n\right)\frac{\partial}{\partial \varphi}\left(\frac{p_n\, x_M^{c3_M}}{\zeta}\right)\right\}, \tag{7.40}$$

$$\sigma_{nM} = -\frac{p_n\, x_M^{c3_M - 1}}{\zeta}\left\{\left[c_{3M}\left(c_{1M} + c_{2M}\right) - 2c_{2M}\right]\left(\frac{1}{2} + \ln x_M\right) - \frac{x_M}{x_n}\left(c_{1M} + 2c_{2M}\ln x_n\right)\right\}, \tag{7.41}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n\, x_M^{c3_M - 1}}{\zeta}\left\{\left(c_{1M} + c_{2M}c_{3M}\right)\left(\frac{1}{2} + \ln x_M\right) - \frac{x_M}{x_n}\left(\frac{c_{1M} - 2c_{2M}}{2} + c_{1M}\ln x_n\right)\right\}, \tag{7.42}$$

$$\sigma_{1M} = \eta_{3M}\, x_n^{c_{3M} - 1} + \frac{\eta_{4M} + \eta_{5M}\ln x_n}{x_n}, \tag{7.43}$$

$$w_M = \left(\frac{p_n}{\zeta}\right)^2\left[\kappa_{2M}\left(\frac{1}{2} + \ln x_M\right)^2 + \kappa_{3M}\, x_M^{2c_{3M}}\right.$$

$$+\kappa_{6M}\, x_M^{c_{3M}} \left(\frac{1}{2}+\ln x_M\right)\Bigg]$$

$$+\frac{\chi_{2M}}{s_{44M}}\left(\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\right\}^2+\Theta^2\left\{\frac{\partial}{\partial o}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\right\}^2\right)$$

$$+\frac{\chi_{3M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\, x_M^{c_{3M}}}{\zeta}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\, x_M^{c_{3M}}}{\zeta}\right)\right]^2\right\}$$

$$+\frac{\chi_{6M}}{s_{44M}}\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\frac{\partial}{\partial\varphi}\left(\frac{p_n\, x_M^{c_{3M}}}{\zeta}\right)\right.$$

$$\left.+\Theta^2\,\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\frac{\partial}{\partial\varphi}\left(\frac{p_n\, x_M^{c_{3M}}}{\zeta}\right)\right\}, \qquad (7.44)$$

$$W_M=4\int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{p_n}{\zeta}\right)^2\left[\Psi_{2M}\left(\frac{1}{2}+\ln x_M\right)^2+\Psi_{3M}\, x_M^{2c_{3M}}\right.$$

$$\left.+\Psi_{6M}\, x_M^{c_{3M}}\left(\frac{1}{2}+\ln x_M\right)\right]\Omega\, d\varphi\, d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{2M}\left(\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\right\}^2\right.$$

$$\left.+\Theta^2\left\{\frac{\partial}{\partial\nu}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\right\}^2\right)\Omega\, d\varphi\, d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{3M}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\, x_M^{c_{3M}}}{\zeta}\right)\right]^2\right.$$

$$\left.+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\, x_M^{c_{3M}}}{\zeta}\right)\right]^2\right\}\Omega\, d\varphi\, d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{6M}\left\{\frac{\partial}{\partial\varphi}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\frac{\partial}{\partial\varphi}\left(\frac{p_n\, x_M^{c_{3M}}}{\zeta}\right)\right.$$

$$\left.+\Theta^2\,\frac{\partial}{\partial v}\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right]\frac{\partial}{\partial\varphi}\left(\frac{p_n\, x_M^{c_{3M}}}{\zeta}\right)\right\}\Omega\, d\varphi\, dv\,, \tag{7.45}$$

where Θ, Ω; x_M; s_{44M}, c_{iM} (i = 1,2,3) and κ_{jM}, χ_{jM}; Φ_{jM}, Ψ_{jM} (j = 2,3,6) are given by Equations (1.13), (1.15), (2.16), (2.21) and (7.13); (7.14), respectively, and ς, ς_i, η_{jM} (i = 1,2; j = 3,4,5; see Equation (7.13)) have the forms

$$\zeta=\frac{\zeta_2}{x_{IN}}\left(\frac{1}{2}+\ln x_M\right)-\zeta_1\, x_M^{c_{3M}-1},\quad \zeta_1=\frac{x_{IM}}{x_{IN}}\left(c_{1M}-2\,c_{2M}\ln x_{IN}\right),$$

$$\zeta_2=\left[c_{3M}\left(c_{1M}+c_{2M}\right)-2\,c_{2M}\right]x_M\, x_{IN}^{c_{3M}-1},$$

$$\eta_{3M}=\frac{p_n\, x_M^{c_{3M}}\left(\gamma_{1M}\, c_{3M}+\gamma_{2M}\right)}{\zeta}$$

$$+\frac{1}{s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi}+\gamma_{4M}\frac{\partial}{\partial v}\right)\left(\frac{p_n\, x_M^{c_{3M}}}{\zeta}\right),$$

$$\eta_{4M}=\frac{p_n\left(2\gamma_{1M}+\gamma_{2M}\right)}{2\zeta}\left(\frac{1}{2}+\ln x_M\right)+$$

$$+\frac{1}{2\,s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi}+\gamma_{4M}\frac{\partial}{\partial v}\right)\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right],$$

$$\eta_{5M}=\frac{p_n\,\gamma_{2M}}{\zeta}\left(\frac{1}{2}+\ln x_M\right)+$$

$$+\frac{1}{2\,s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi}+\gamma_{4M}\frac{\partial}{\partial v}\right)\left[\frac{p_n}{\zeta}\left(\frac{1}{2}+\ln x_M\right)\right], \tag{7.46}$$

The normal stress p_n is given by Equation (4.6). With regard to Equa-tion (7.38), the coefficient ρ_M in Equation (4.6) is derived as

$$\rho_M = \frac{1}{\zeta}\left[\left(\frac{1}{2}+\ln x_M\right)x_{IN}^{c3_M-1} - \frac{x_M^{c3_M}}{x_{IN}}\left(\frac{1}{2}+\ln x_{IN}\right)\right]. \quad (7.47)$$

Conditions $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} \neq 0$. With regard to Equations (2.30), (4.1)-(4.3), (7.4)-(7.12), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta}\left[\zeta_1\left(\frac{1}{3}-\ln x_n\right)+\zeta_2\, c_{3M} x_n^{c_{3M}-1}+\frac{\zeta_3}{x_n}\right], \quad (7.48)$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = \frac{u_{nM}}{x_n}$$

$$= -\frac{p_n}{\zeta}\left[\zeta_1\left(\frac{4}{3}-\ln x_n\right)+\zeta_2\, x_n^{c_{3M}-1}+\frac{\zeta_3}{x_n}\left(\frac{1}{2}+\ln x_n\right)\right], \quad (7.49)$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = -\left[\left(\frac{4}{3}-\ln x_n\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)+x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right)\right)$$

$$+\frac{1}{x_n}\left(\frac{1}{2}+\ln x_n\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right)\Bigg], \quad (7.50)$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = -\Theta\left[\left(\frac{4}{3}-\ln x_n\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)+x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right)\right)$$

$$+\frac{1}{x_n}\left(\frac{1}{2}+\ln x_n\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right)\Bigg], \quad (7.51)$$

$$\sigma_{nM} = -\frac{p_n}{\zeta}\Bigg\{\zeta_1\left[\frac{c_{1M}-7c_{2M}}{3}-(c_{1M}-c_{2M})\ln x_n\right]$$

$$+\zeta_2\left[(c_{1M}+c_{2M})c_{3M}-2c_{2M}\right]x_n^{c_{3M}-1}+\frac{\zeta_3\left(c_{1M}-2c_{2M}\ln x_n\right)}{x_n}\Bigg\}, \quad (7.52)$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta}\Bigg\{\zeta_1\left[\frac{4c_{1M}-c_{2M}}{3}-(c_{1M}-c_{2M})\ln x_n\right]$$

$$+\zeta_2\left(c_{1M}-c_{2M}c_{3M}\right)x_n^{c_{3M}-1}+\zeta_3\left(\frac{c_{1M}-2c_{2M}}{2}+c_{1M}\ln x_n\right)\Bigg\},\qquad(7.53)$$

$$\sigma_{1M}=\eta_{1M}+\eta_{2M}\ln x_n+\eta_{3M}x_n^{c_{3M}-1}+\frac{\eta_{4M}+\eta_{4M}\ln x_n}{x_n},\qquad(7.54)$$

$$w_M=\left(\frac{p_n}{\zeta}\right)^2(\kappa_{1M}\zeta_1^2+\kappa_{2M}\zeta_2^2+\kappa_{3M}\zeta_3^2$$
$$+\kappa_{4M}\zeta_1\zeta_2+\kappa_{5M}\zeta_1\zeta_3+\kappa_{6M}\zeta_2\zeta_3)$$
$$+\frac{\chi_{1M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\right]^2\right\}$$
$$+\frac{\chi_{2M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right)\right]^2\right\}$$
$$+\frac{\chi_{3M}}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]^2\right\}$$
$$+\frac{\chi_{4M}}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right)+\Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right)\right]$$
$$+\frac{\chi_{5M}}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right)+\Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]$$
$$+\frac{\chi_{6M}}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right)+\Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right],$$
(7.55)

$$W_M=4\int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{p_n}{\zeta}\right)^2\Big(\Phi_{1M}\zeta_1^2+\Phi_{2M}\zeta_2^2+\Phi_{3M}\zeta_3^2+\Phi_{4M}\zeta_1\zeta_2$$
$$+\Phi_{5M}\zeta_1\zeta_3+\Phi_{6M}\zeta_2\zeta_3\Big)\Omega\,d\varphi d\nu$$
$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{1M}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\right]^2\right\}\Omega d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{2M}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right)\right]^2\right\}\Omega d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{3M}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]^2\right\}\Omega d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{4M}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right)\right]\Omega d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{4M}\,\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right)\right]\Omega d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{5M}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]\Omega d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{5M}\,\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_1}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]\Omega d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{6M}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_2}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]\Omega d\varphi d\nu$$

$$+\frac{4}{s_{44M}}\int_0^{\pi/2}\int_0^{\pi/2}\Psi_{6M}\,\Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_2}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\zeta_3}{\zeta}\right)\right]\Omega d\varphi d\nu\,, \quad (7.56)$$

where Θ, Ω; x_M; s_{44M}, c_{iM} (i = 1,2,3) and κ_{jM}, χ_{jM}; Φ_{jM}, Ψ_{jM} (j = 1,…,6) are given by Equations (1.13), (1.15), (2.16), (2.21) and (7.13); (7.14), respectively, and ς, ς_i (i = 1,2,3), η_{iM} (i = 1,…,5; see Equation (7.13)) have the forms

$$\zeta_1 = x_M^{c_{3M}-1}\left[c_{3M}\left(\frac{1}{2}+\ln x_M\right)-1\right],$$

$$\zeta_2 = \frac{1}{4} - \ln x_M - \left(\frac{1}{2}+\ln x_M\right)\left(\frac{1}{3}-\ln x_M\right),$$

$$\zeta_3 = x_M^{c_{3M}}\left[\frac{1}{3} - \ln x_M - c_{3M}\left(\frac{4}{3} - \ln x_M\right)\right],$$

$$\zeta = x_M^{c_{3M}-1}\left[\frac{c_{1M} - 7c_{2M}}{3} - (c_{1M} - c_{2M})\ln x_{IN}\right]\left[1 - c_{3M}\left(\frac{1}{2} + \ln x_M\right)\right]$$

$$+ x_{IN}^{c_{3M}-1}\left[(c_{1M} + c_{2M})\, c_{3M} - 2c_{2M}\right]\left(\frac{1}{2} + \ln x_M\right)\left(\frac{1}{3} - \ln x_M\right)$$

$$\times\left[\frac{4}{3} - \ln x_M + \left(\frac{1}{2} + \ln x_M\right)\left(\frac{1}{3} - \ln x_M\right)\right]$$

$$+ \frac{x_{IM}^{c_{3M}}}{x_{IN}}(c_{1M} + c_{2M}\ln x_M)\left[\frac{1}{3} - \ln x_M + c_{3M}\left(\frac{4}{3} - \ln x_M\right)\right],$$

$$\eta_{1M} = -\frac{p_n \zeta_1(\gamma_{1M} + 4\gamma_{2M})}{3\zeta} - \frac{4}{3s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi} + \gamma_{4M}\frac{\partial}{\partial\nu}\right)\left(\frac{p_n\zeta_1}{\zeta}\right),$$

$$\eta_{2M} = \frac{p_n \zeta_1(\gamma_{1M} + \gamma_{2M})}{\zeta} + \frac{1}{s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi} + \gamma_{4M}\frac{\partial}{\partial\nu}\right)\left(\frac{p_n\zeta_1}{\zeta}\right),$$

$$\eta_{3M} = -\frac{p_n \zeta_2(\gamma_{1M} c_{3M} + \gamma_{2M})}{\zeta} - \frac{1}{s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi} + \gamma_{4M}\frac{\partial}{\partial\nu}\right)\left(\frac{p_n\zeta_2}{\zeta}\right)$$

$$\eta_{4M} = -\frac{p_n \zeta_3(2\gamma_{1M} + \gamma_{2M})}{\zeta} - \frac{1}{s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi} + \gamma_{4M}\frac{\partial}{\partial\nu}\right)\left(\frac{p_n\zeta_3}{\zeta}\right),$$

$$\eta_{5M} = -\frac{p_n \zeta_3\gamma_{2M}}{\zeta} - \frac{1}{s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi} + \gamma_{4M}\frac{\partial}{\partial\nu}\right)\left(\frac{p_n\zeta_3}{\zeta}\right). \tag{7.57}$$

The normal stress p_n is given by Equation (4.6). With regard to Equa-tion (6.50), the coefficient ρ_M in Equation (4.6) is derived as

$$\rho_M = \frac{1}{\zeta}\left[\zeta_1\left(\frac{4}{3} - \ln x_{IN}\right) + \zeta_2\, x_{IN}^{c_{3M}-1} + \frac{\zeta_3}{x_{IN}}\left(\frac{1}{2} + \ln x_{IN}\right)\right]. \tag{7.58}$$

CHAPTER 8

MATHEMATICAL MODEL 4

8.1 Mathematical procedure 4

The differential equation (2.26) is derived as

$$U_n = -s_{44}(c_1 + c_2)\left(x_n^2 \frac{\partial^2 u_n}{\partial x_n^2} + 2x_n \frac{\partial u_n}{\partial x_n} - 2u_n \right), \tag{8.1}$$

where s_{44}, c_i (i = 1,2), $U_n = U_n(x_n, \varphi, \theta)$ are given by Equation (2.16), (2.21), (2.28), respectively. If $x_n[\partial Eq.(8.1)/\partial x_n]$, then we get

$$x_n \frac{\partial U_n}{\partial x_n} = -s_{44}(c_1 + c_2)\left(x_n^3 \frac{\partial^3 u_n}{\partial x_n^3} + 4x_n^2 \frac{\partial^2 u_n}{\partial x_n^2} \right). \tag{8.2}$$

If Equations (8.1), (8.2) are substituted to Equation (2.27), then we get

$$x_n^3 \frac{\partial^3 u_n}{\partial x_n^3} + (4 - c_3)x_n^2 \frac{\partial^2 u_n}{\partial x_n^2} + 2c_3\left(u_n - x_n \frac{\partial u_n}{\partial x_n} \right) = 0. \tag{8.3}$$

If $u_n = x_n^{\lambda}$, then, with respect to Equation (8.3), we get

$$u_n = C_1 x_n + C_2 x_n^{c_3} + \frac{C_3}{x_n^2}, \tag{8.4}$$

where $c_3 < 0$ is given by Equation (2.21), and the integration constants C_1, C_2, C_3 are determined by the boundary conditions in Chapter 4. With regard to Equation (2.1)-(2.4), (2.17)-(2.20), (2.29), (8.4), we get

$$\varepsilon_n = C_1 + C_2\, c_3\, x_n^{c_3-1} - \frac{2\,C_3}{x_n^3}\ , \tag{8.5}$$

$$\varepsilon_\varphi = \varepsilon_\nu = \frac{u_n}{x_n} = C_1 + C_2\, x_n^{c_3-1} + \frac{C_3}{x_n^3}\ , \tag{8.6}$$

$$\varepsilon_{n\varphi} = s_{44}\,\sigma_{n\varphi} = \frac{\partial C_1}{\partial\varphi} + x_n^{c_3-1}\frac{\partial C_2}{\partial\varphi} + \frac{1}{x_n^3}\frac{\partial C_3}{\partial\varphi}\,, \tag{8.7}$$

$$\varepsilon_{n\theta} = s_{44}\,\sigma_{n\theta} = \Theta\left[\ \frac{\partial C_1}{\partial\nu} + x_n^{c_3-1}\frac{\partial C_2}{\partial\nu} + \frac{1}{x_n^3}\frac{\partial C_3}{\partial\nu}\right], \tag{8.8}$$

$$\sigma_n = C_1\left(c_1 - c_2\right) + C_2\left[\left(c_1 + c_2\right)c_3 - 2c_2\right]x_n^{c_3-1} - \frac{2\,C_3\left(c_1 + 2c_2\right)}{x_n^3}\,, \tag{8.9}$$

$$\sigma_\varphi = \sigma_\nu = C_1\left(c_1 - c_2\right) + C_2\left(c_1 - c_2\,c_3\right)x_n^{c_3-1} + \frac{C_3\left(c_1 + 2c_2\right)}{x_n^3}\,, \tag{8.10}$$

$$\sigma_1 = \eta_1 + \eta_2\, x_n^{c_3-1} + \frac{\eta_3}{x_n^3}\,, \tag{8.11}$$

$$w = \kappa_1 + \kappa_2\, x_n^{2(c_3-1)} + \frac{\kappa_3}{x_n^6} + \kappa_4\, x_n^{c_3-1} + \frac{\kappa_5}{x_n^3} + \kappa_6\, x_n^{c_3-4}\,, \tag{8.12}$$

where Θ is given by Equation (1.15); and η_i, κ_j (i = 1,2,3; j = 1,…,6) are derived as

$$\eta_1 = C_1\left(\gamma_1 + \gamma_2\right) + \frac{1}{s_{44}}\left(\gamma_3\frac{\partial C_1}{\partial\varphi} + \gamma_4\frac{\partial C_1}{\partial\nu}\right),$$

$$\eta_2 = C_2\left(\gamma_1\, c_3 + \gamma_2\right) + \frac{1}{s_{44}}\left(\gamma_3\frac{\partial C_2}{\partial\varphi} + \gamma_4\frac{\partial C_2}{\partial\nu}\right),$$

$$\eta_3 = C_3\left(\gamma_2 - 2\gamma_1\right) + \frac{1}{s_{44}}\left(\gamma_3\frac{\partial C_3}{\partial\varphi} + \gamma_4\frac{\partial C_3}{\partial\nu}\right),$$

$$\kappa_1 = \frac{3(c_1 - c_2)C_1^2}{2} + \frac{1}{s_{44}}\left[\left(\frac{\partial C_1}{\partial \varphi}\right)^2 + \Theta^2\left(\frac{\partial C_1}{\partial \nu}\right)^2\right],$$

$$\kappa_2 = \left[\frac{(c_1 + c_2)c_3^2}{2} + c_1 - 2c_2 c_3\right] C_2^2 + \frac{1}{s_{44}}\left[\left(\frac{\partial C_2}{\partial \varphi}\right)^2 + \Theta^2\left(\frac{\partial C_2}{\partial \nu}\right)^2\right],$$

$$\kappa_3 = 3(c_1 + 2c_2)C_3^2 + \frac{1}{s_{44}}\left[\left(\frac{\partial C_3}{\partial \varphi}\right)^2 + \Theta^2\left(\frac{\partial C_3}{\partial \nu}\right)^2\right],$$

$$\kappa_4 = (c_1 - c_2)(2 + c_3)C_1 C_2 + \frac{1}{s_{44}}\left(\frac{\partial C_1}{\partial \varphi}\frac{\partial C_2}{\partial \varphi} + \Theta^2 \frac{\partial C_1}{\partial \nu}\frac{\partial C_2}{\partial \nu}\right),$$

$$\kappa_5 = \frac{2}{s_{44}}\left(\frac{\partial C_1}{\partial \varphi}\frac{\partial C_3}{\partial \varphi} + \Theta^2 \frac{\partial C_1}{\partial \nu}\frac{\partial C_3}{\partial \nu}\right),$$

$$\kappa_6 = [2c_2(1 - c_3) - c_1]C_2 C_3 + \frac{2}{s_{44}}\left(\frac{\partial C_2}{\partial \varphi}\frac{\partial C_3}{\partial \varphi} + \Theta^2 \frac{\partial C_2}{\partial \nu}\frac{\partial C_3}{\partial \nu}\right). \quad (8.13)$$

8.2 Cell Matrix

The integration constants C_{1M}, C_{2M}, C_{3M} are determined by Equations (4.1), (4.2) or (4.1)-(4.3), and then the following combina-tions are considered: $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} = 0$; $C_{1M} \neq 0$, $C_{3M} \neq 0$, $C_{2M} = 0$; $C_{2M} \neq 0$, $C_{3M} \neq 0$, $C_{1M} = 0$; $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} \neq 0$. Finally, such a combination is considered to exhibit mininum potential energy $W_p = W_{IN} + W_M$ (see Equation (2.30)).

Conditions $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} = 0$. With regard to Equations (2.30), (4.1), (4.2), (8.4)-(8.12), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta}\left[1 - c_{3M}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right], \quad (8.14)$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = \frac{u_{nM}}{x_n} = -\frac{p_n}{\zeta}\left[1 - \left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right], \quad (8.15)$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = -\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right) - x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\right], \quad (8.16)$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = -\,\Theta\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right) - x_n^{c_{3M}-1}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\right], \quad (8.17)$$

$$\sigma_{nM} = -\frac{p_n}{\zeta}\left\{c_{1M} - c_{2M} - \left[\left(c_{1M} + c_{2M}\right)c_{3M} - 2c_{2M}\right]\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right\}, \quad (8.18)$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta}\left\{c_{1M} - c_{2M} - \left(c_{1M} + c_{2M}\,c_{3M}\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right\}, \quad (8.19)$$

$$\sigma_{1M} = \eta_{1M} + \eta_{2M}\,x_n^{c_{3M}-1}, \quad (8.20)$$

$$w_M = \kappa_{1M} + \kappa_{2M}\,x_n^{2\left(c_{3M}-1\right)} + \kappa_{4M}\,x_n^{c_{3M}-1}, \quad (8.21)$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\kappa_{1M}}{3}\left(x_M^3 - x_{IN}^3\right) + \frac{\kappa_{2M}}{2c_{3M}+1}\left(x_M^{2c_{3M}+1} - x_{IN}^{2c_{3M}+1}\right) + \frac{\kappa_{4M}}{c_{3M}+1}\left(x_M^{c_{3M}+1} - x_{IN}^{c_{3M}+1}\right)\right]\Omega\, d\varphi\, d\nu, \quad (8.22)$$

where Θ, Ω; x_{IN}, x_M; s_{44M}, c_{iM} (i = 1,2,3) are given by Equations (1.13); (1.15); (2.16), (2.21), respectively, and ς, η_{jM}, κ_{jM}, (i = 1,2; j = 1,2,4; see Equation (8.13)) have the forms

$$\zeta = c_{1M} - c_{2M} - \left[\left(c_{1M} + c_{2M}\right)c_{3M} - 2c_{2M}\right]\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1},$$

$$\eta_{1M} = -\left[\frac{p_n(\gamma_{1M}+\gamma_{2M})}{\zeta} + \frac{1}{s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi} + \gamma_{4M}\frac{\partial}{\partial\nu}\right)\left(\frac{p_n}{\zeta}\right)\right],$$

$$\eta_{2M} = \frac{p_n(\gamma_{1M}\,c_{3M}+\gamma_{2M})}{\zeta\,x_M^{c_{3M}-1}} + \frac{1}{s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi} + \gamma_{4M}\frac{\partial}{\partial\nu}\right)\left(\frac{p_n}{\zeta\,x_M^{c_{3M}-1}}\right)$$

$$\kappa_{1M} = \frac{3(c_{1M}-c_{2M})}{2}\left(\frac{p_n}{\zeta}\right)^2 + \frac{1}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right]^2\right\}$$

$$\kappa_{2M} = \left[\frac{(c_{1M}-c_{2M})c_{3M}^2}{2} + c_{1M} - 2c_{2M}\,c_{3M}\right]\left(\frac{p_n}{\zeta\,x_M^{c_{3M}-1}}\right)^2$$

$$+\frac{1}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\,x_M^{c_{3M}-1}}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\,x_M^{c_{3M}-1}}\right)\right]^2\right\},$$

$$\kappa_{4M} = \frac{(c_{2M}-c_{1M})(2+c_{3M})}{x_M^{c_{3M}-1}}\left(\frac{p_n}{\zeta}\right)^2$$

$$-\frac{2}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\,x_M^{c_{3M}-1}}\right) + \Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\,x_M^{c_{3M}-1}}\right)\right]. \tag{8.23}$$

The normal stress p_n is given by Equation (4.6). With regard to Equa-tion (8.15), the coefficient ρ_M in Equation (4.6) is derived as

$$\rho_M = \frac{1}{\zeta}\left[1-\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1}\right]. \tag{8.24}$$

Conditions $C_{1M} \neq 0$, $C_{3M} \neq 0$, $C_{2M} = 0$. With regard to Equations (2.30), (4.1), (4.2), (8.4)-(8.12), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta}\left[1-3\left(\frac{x_M}{x_n}\right)^3\right], \tag{8.25}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = \frac{u_{nM}}{x_n} = -\frac{p_n}{\zeta}\left[1 - \left(\frac{x_M}{x_n}\right)^3\right], \tag{8.26}$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = -\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right) - \frac{1}{x_n^3}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,x_M^3}{\zeta}\right)\right], \tag{8.27}$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = -\,\Theta\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right) - \frac{1}{x_n^3}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,x_M^3}{\zeta}\right)\right], \tag{8.28}$$

$$\sigma_{nM} = -\frac{p_n}{\zeta}\left[c_{1M} - c_{2M} + 2\left(c_{1M} + 2c_{2M}\right)\left(\frac{x_M}{x_n}\right)^3\right], \tag{8.29}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta}\left[c_{1M} - c_{2M} - \left(c_{1M} + 2c_{2M}\right)\left(\frac{x_M}{x_n}\right)^3\right], \tag{8.30}$$

$$\sigma_{1M} = \eta_{1M} + \frac{\eta_{3M}}{x_n^3}, \tag{8.31}$$

$$w_M = \kappa_{1M} + \frac{\kappa_{3M}}{x_n^6} + \frac{\kappa_{5M}}{x_n^3}, \tag{8.32}$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\kappa_{1M}}{3}\left(x_M^3 - x_{IN}^3\right) + \frac{\kappa_{3M}}{3}\left(\frac{1}{x_{IN}^3} - \frac{1}{x_M^3}\right) + \kappa_{5M}\ln\left(\frac{x_M}{x_{IN}}\right)\right]\Omega\, d\varphi\, d\nu, \tag{8.33}$$

where Θ, Ω; x_{IN}, x_M; s_{44M}, c_{iM} (i = 1,2,3) are given by Equations (1.13); (1.15); (2.16), (2.21), respectively, and ς, η_{3M}, κ_{iM} (i = 3,5; see Equation (8.13)) have the forms

$$\zeta = c_{1M} - c_{2M} + 2(c_{1M} + 2c_{2M})\left(\frac{x_M}{x_{IN}}\right)^2 ,$$

$$\eta_{3M} = \frac{p_n x_M^3 (\gamma_{2M} - 2\gamma_{1M})}{\zeta} + \frac{1}{s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi} + \gamma_{4M}\frac{\partial}{\partial\nu}\right)\left(\frac{p_n x_M^3}{\zeta}\right),$$

$$\kappa_{3M} = 3(c_{1M} + 2c_{2M})\left(\frac{p_n x_M^3}{\zeta}\right)^2$$

$$+ \frac{1}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{\zeta}\right)\right]^2\right\},$$

$$\kappa_{5M} = -\frac{2}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{\zeta}\right)\right.$$

$$\left. + \Theta^2\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{\zeta}\right)\right]. \tag{8.34}$$

The coefficients η_{1M}, κ_{1M} are given by Equation (8.23), where ς in Equation (8.23) is given by Equation (8.34). The normal stress p_n is given by Equation (4.6). With regard to Equation (8.26), the coefficient ρ_M in Equation (4.6) is derived as

$$\rho_M = \frac{1}{\zeta}\left[1 - \left(\frac{x_M}{x_{IN}}\right)^3\right]. \tag{8.35}$$

Conditions $C_{2M} \neq 0$, $C_{3M} \neq 0$, $C_{1M} = 0$. With regard to Equations (2.30), (4.1), (4.2), (8.4)-(8.12), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta}\left[c_{3M}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} - 2\left(\frac{x_M}{x_n}\right)^3\right], \tag{8.36}$$

$$\varepsilon_{\varphi M}=\varepsilon_{\theta M}=\frac{u_{nM}}{x_n}=-\frac{p_n}{\zeta}\left[\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}-\left(\frac{x_M}{x_n}\right)^3\right], \tag{8.37}$$

$$\varepsilon_{n\varphi M}=s_{44M}\,\sigma_{n\varphi M}=-\left[x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\right.$$
$$\left.-\frac{1}{x_n^3}\frac{\partial}{\partial\varphi}\left(\frac{p_n\, x_M^3}{\zeta}\right)\right], \tag{8.38}$$

$$\varepsilon_{n\theta M}=s_{44M}\,\sigma_{n\theta M}=-\Theta^2\left[x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\right.$$
$$\left.-\frac{1}{x_n^3}\frac{\partial}{\partial\nu}\left(\frac{p_n\, x_M^3}{\zeta}\right)\right], \tag{8.39}$$

$$\sigma_{nM}=-\frac{p_n}{\zeta}\left\{\left[c_{3M}\left(c_{1M}+c_{2M}\right)-2c_{2M}\right]\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right.$$
$$\left.-2\left(c_{1M}+2c_{2M}\right)\left(\frac{x_M}{x_n}\right)^3\right\}, \tag{8.40}$$

$$\sigma_{\varphi M}=\sigma_{\theta M}=-\frac{p_n}{\zeta}\left[\left(c_{1M}+c_{2M}c_{3M}\right)\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}\right.$$
$$\left.+\left(c_{1M}+2c_{2M}\right)\left(\frac{x_M}{x_n}\right)^3\right], \tag{8.41}$$

$$\sigma_{1M}=\eta_{2M}\,x_n^{c_{3M}-1}+\frac{\eta_{4M}}{x_n^3}, \tag{8.42}$$

$$w_M = \kappa_{2M}\, x_n^{2(c_{3M}-1)} + \frac{\kappa_{3M}}{x_n^6} + \kappa_{6M}\, x_n^{c_{3M}-4}\,, \tag{8.43}$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\kappa_{2M}}{2c_{3M}+1}\left(x_M^{2c_{3M}+1} - x_{IN}^{2c_{3M}+1}\right) + \frac{\kappa_{3M}}{3}\left(\frac{1}{x_{IN}^3} - \frac{1}{x_M^3}\right)\right.$$
$$\left. + \frac{\kappa_{6M}}{c_{3M}-1}\left(x_M^{c_{3M}-1} - x_{IN}^{c_{3M}-1}\right)\right]\Omega\, d\varphi\, d\nu\,, \tag{8.44}$$

where Θ, Ω; x_{IN}, x_M; s_{44M}, c_{iM} (i = 1,2,3) are given by Equations (1.13), (1.15), (2.16), (2.21), respectively, and ς, κ_{6M} (see Equation (7.13)) have the forms

$$\zeta = \left\{\left[c_{3M}\left(c_{1M}+c_{2M}\right) - 2c_{2M}\right]\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}+2} + 2\left(c_{1M}+2c_{2M}\right)\right\}\left(\frac{x_M}{x_{IN}}\right)^3,$$

$$\kappa_{6M} = -\frac{\left[2c_{2M}\left(1-c_{3M}\right)-c_{1M}\right]}{x_M^{c_{3M}-4}}\left(\frac{p_n}{\zeta}\right)^2$$
$$-\frac{2}{s_{44M}}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\, x_M^3}{\zeta}\right)$$
$$-\frac{2\Theta^2}{s_{44M}}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\, x_M^3}{\zeta}\right). \tag{8.45}$$

The coefficients η_{2M}, κ_{2M} and η_{3M}, κ_{3M} are given by Equations (8.23) and (8.34), where ς in Equations (8.23), (8.34) is given by Equation (8.45).

The normal stress p_n is given by Equation (4.6). With regard to Equation (8.37), the coefficient ρ_M in Equation (4.6) is derived as

$$\rho_M = \frac{1}{\zeta}\left[\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1} - \left(\frac{x_M}{x_{IN}}\right)^3\right]. \tag{8.46}$$

Conditions $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{3M} \neq 0$. With regard to Equations (2.30), (4.1)-(4.3), (8.4)-(8.12), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta}\left\{1-\frac{1}{c_{3M}+2}\left[3c_{3M}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} - 2(c_{3M}-1)\left(\frac{x_M}{x_n}\right)^3\right]\right\}, \tag{8.47}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = \frac{u_{nM}}{x_n} = -\frac{p_n}{\zeta}\left\{1-\frac{1}{c_{3M}+2}\left[3\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} + (c_{3M}-1)\left(\frac{x_M}{x_n}\right)^3\right]\right\}, \tag{8.48}$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = -\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right) - \frac{3}{c_{3M}+2}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)x_n^{c_{3M}-1} - \frac{c_{3M}-1}{(c_{3M}+2)x_n^3}\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{\zeta}\right)\right], \tag{8.49}$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = -\Theta^2\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right) - \frac{3}{c_{3M}+2}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)x_n^{c_{3M}-1} - \frac{c_{3M}-1}{(c_{3M}+2)x_n^3}\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M^3}{\zeta}\right)\right], \tag{8.50}$$

$$\sigma_{nM} = -\frac{p_n}{\zeta}\left\{c_{1M} - c_{2M} - \frac{3\left[(c_{1M}+c_{2M})c_{3M} - 2c_{2M}\right]}{c_{3M}+2}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} + \frac{2(c_{1M}+2\,c_{2M})}{c_{3M}+2}\left(\frac{x_M}{x_n}\right)^3\right\}, \tag{8.51}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta}\left\{c_{1M} - c_{2M} - \frac{3\left(c_{1M} - c_{2M}\,c_{3M}\right)}{c_{3M}+2}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} + \frac{c_{1M}+2\,c_{2M}}{c_{3M}+2}\left(\frac{x_M}{x_n}\right)^3\right\}, \tag{8.52}$$

$$\sigma_{1M} = \eta_{1M} + \eta_{2M}\,x_n^{c_{3M}-1} + \frac{\eta_{3M}}{x_n^3}, \tag{8.53}$$

$$w_M = \kappa_{1M} + \kappa_{2M}\,x_n^{2\left(c_{3M}-1\right)} + \frac{\kappa_{3M}}{x_n^6} + \kappa_{4M}\,x_n^{c_{3M}-1} + \frac{\kappa_{5M}}{x_n^3} + \kappa_{6M}\,x_n^{c_{3M}-4}, \tag{8.54}$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\kappa_{1M}}{3}\left(x_M^3 - x_{IN}^3\right) + \frac{\kappa_{2M}}{2c_{3M}+1}\left(x_M^{2c_{3M}+1} - x_{IN}^{2c_{3M}+1}\right) + \frac{\kappa_{3M}}{3}\left(\frac{1}{x_{IN}^3} - \frac{1}{x_M^3}\right) + \frac{\kappa_{4M}}{c_{3M}+2}\left(x_M^{c_{3M}+2} - x_{IN}^{c_{3M}+2}\right) + \kappa_{3M}\ln\left(\frac{x_M}{x_{IN}}\right) + \frac{\kappa_{6M}}{c_{3M}-1}\left(x_M^{c_{3M}-1} - x_{IN}^{c_{3M}-1}\right)\right]\Omega\,d\varphi\,d\nu, \tag{8.55}$$

where Θ, Ω; x_{IN}, x_M; s_{44M}, c_{iM} (i = 1,2,3) are given by Equations (1.13), (1.15), (2.16), (2.21), respectively, and ς, η_{iM}, κ_{jM}, (i = 2,3; j = 2,…,6; see Equation (7.13)) have the forms

$$\zeta = c_{1M} - c_{2M} + \frac{1}{c_{3M}+2}\left(\frac{x_M}{x_{IN}}\right)^3\left\{2\left(c_{3M}-1\right)\left(c_{1M}+2c_{2M}\right) - 3\left[\left(c_{1M}+c_{2M}\right)c_{3M} - 2c_{2M}\right]\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}+2}\right\},$$

$$\eta_{2M} = \frac{3}{c_{3M}+2}\left\{\frac{p_n\left(\gamma_{1M}\,c_{3M}+\gamma_{2M}\right)}{\zeta\, x_M^{c_{3M}-1}}\right.$$

$$\left.+\frac{1}{s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi}+\gamma_{4M}\frac{\partial}{\partial\nu}\right)\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\right\},$$

$$\eta_{3M} = \frac{c_{3M}-1}{c_{3M}+2}\left\{\frac{p_n\, x_M^3\left(\gamma_{2M}-2\gamma_{2M}\right)}{\zeta}\right.$$

$$\left.+\frac{1}{s_{44M}}\left(\gamma_{3M}\frac{\partial}{\partial\varphi}+\gamma_{4M}\frac{\partial}{\partial\nu}\right)\left(\frac{p_n\, x_M^3}{\zeta}\right)\right\},$$

$$\kappa_{2M} = \left(\frac{3}{c_{3M}+2}\right)^2$$

$$\times\left(\left[\frac{\left(c_{1M}-c_{2M}\right)c_{3M}^2}{2}+c_{1M}-2c_{2M}\,c_{3M}\right]\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)^2\right.$$

$$\left.+\frac{1}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\right]^2\right\}\right),$$

$$\kappa_{3M} = \left(\frac{c_{3M}-1}{c_{3M}+2}\right)^2\left(3\left(c_{1M}+2c_{2M}\right)\left(\frac{p_n\, x_M^3}{\zeta}\right)^2\right.$$

$$\left.+\frac{1}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\, x_M^3}{\zeta}\right)\right]^2+\Theta^2\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\, x_M^3}{\zeta}\right)\right]^2\right\}\right),$$

$$\kappa_{4M} = \frac{3\left(c_{2M}-c_{1M}\right)}{x_M^{c_{3M}-1}}\left(\frac{p_n}{\zeta}\right)^2$$

$$-\frac{6}{s_{44M}\left(c_{3M}+2\right)}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\right].$$

$$+\Theta^2 \frac{\partial}{\partial \nu}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial \nu}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\Bigg]\Bigg],$$

$$\kappa_{5M} = \frac{2(1-c_{3M})}{s_{44M}(c_{3M}+2)}\left[\frac{\partial}{\partial \varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial \varphi}\left(\frac{p_n x_M^3}{\zeta}\right)\right.$$

$$\left.+\Theta^2 \frac{\partial}{\partial \varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial \varphi}\left(\frac{p_n x_M^3}{\zeta}\right)\right],$$

$$\kappa_{6M} = \frac{3(c_{3M}-1)[2c_{2M}(1-c_{3M})-c_{1M}]}{x_M^{c_{3M}-4}}\left[\frac{p_n}{\zeta(c_{3M}+2)}\right]^2$$

$$+\frac{6(c_{3M}-1)}{s_{44M}(c_{3M}+2)^2}\frac{\partial}{\partial \varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\frac{\partial}{\partial \varphi}\left(\frac{p_n x_M^3}{\zeta}\right)$$

$$+\frac{6\Theta^2(c_{3M}-1)}{s_{44M}(c_{3M}+2)^2}\frac{\partial}{\partial \varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\frac{\partial}{\partial \varphi}\left(\frac{p_n x_M^3}{\zeta}\right). \qquad (8.56)$$

The coefficients η_{1M}, κ_{1M} are given by Equation (8.23), where ς in Equation (8.23) is given by Equation (8.56). The normal stress p_n is given by Equation (4.6). With regard to Equation (8.48), the coefficient ρ_M in Equation (4.6) is derived as

$$\rho_M = \frac{1}{\zeta}\left\{1-\frac{1}{c_{3M}+2}\left[3\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1}+(c_{3M}-1)\left(\frac{x_M}{x_{IN}}\right)^3\right]\right\} \qquad (8.57)$$

8.3 Ellipsoidal Inclusion

In case of the ellipsoidal inclusion, we get $C_{2IN} = C_{3IN} = 0$, otherwise $(u_{nIN})_{x_n=0} = \pm\infty$, $(\varepsilon_{nIN})_{x_n=0} = \pm\infty$, $(\sigma_{nIN})_{x_n=0} = \pm\infty$. With regard to Equations (2.30), (4.4), (4.5), (8.4)-(8.12), we get

$$\varepsilon_{nIN} = \varepsilon_{\varphi IN} = \varepsilon_{\theta IN} = \frac{u_{nIN}}{x_n} = -p_n \rho_{IN}, \qquad (8.58)$$

$$\varepsilon_{n\varphi IN} = s_{44IN}\, \sigma_{n\varphi IN} = -\, \rho_{IN} \frac{\partial p_n}{\partial \varphi}\,, \tag{8.59}$$

$$\varepsilon_{n\varphi IN} = s_{44IN}\, \sigma_{n\varphi IN} = -\, \Theta\, \rho_{IN} \frac{\partial p_n}{\partial \nu}\,, \tag{8.60}$$

$$\sigma_{nIN} = \sigma_{\varphi IN} = \sigma_{\theta IN} = -p_n\,, \tag{8.61}$$

$$\sigma_{1IN} = -p_n\, \rho_{IN} \left[p_n (\gamma_1 + \gamma_2) + \frac{1}{s_{44IN}} \left(\gamma_3 \frac{\partial p_n}{\partial \varphi} + \gamma_4 \frac{\partial p_n}{\partial \nu} \right) \right], \tag{8.62}$$

$$w_{IN} = \rho_{IN}^2 \left\{ \frac{3 p_n^2}{2 \rho_{IN}} + \frac{2}{s_{44IN}} \left[\left(\frac{\partial p_n}{\partial \varphi} \right)^2 + \Theta^2 \left(\frac{\partial p_n}{\partial \nu} \right)^2 \right] \right\}, \tag{8.63}$$

$$W_M = \frac{4}{3} \int_0^{\pi/2} \int_0^{\pi/2} x_{IN}^3\, w_{IN}\, \Omega\, d\varphi\, d\nu\,, \tag{8.64}$$

where Θ, Ω; x_{IN}, $s_{44\mathrm{M}}$ are given by Equations (1.13), (1.15), (2.16), respectively. The normal stress p_{n} is given by Equation (4.6). With regard to Equation (8.59), the coefficient ρ_{IN} in Equation (4.6) is derived as

$$\rho_{IN} = \frac{1 - 2\, \mu_{nIN}}{E_{IN}}\,. \tag{8.66}$$

CHAPTER 9

MATHEMATICAL MODEL 5

9.1 Mathematical procedure 5

If the mathematical procedures ∂Eq.(2.27)/∂x_n, Eq.(6.2)/x_n are per-formed, then Equations (2.27), (6.2) are transformed to the forms

$$x_n \frac{\partial^2 U_n}{\partial x_n^2} + (1 - c_1) \frac{\partial U_n}{\partial x_n} = 0\,, \tag{9.1}$$

$$\frac{\partial U_n}{\partial x_n} = - s_{44}(c_1 + c_2)\left(x_n^2 \frac{\partial^3 u_n}{\partial x_n^3} + 4x_n \frac{\partial^2 u_n}{\partial x_n^2} \right), \tag{9.2}$$

where s_{44}, c_i (i = 1,2), $U_n = U_n\,(x_n,\varphi,\theta)$ are given by Equation (2.16), (2.21), (2.28), respectively, and $c_3 < 0$. If the mathematical procedure ∂Eq.(9.2)/∂x_n is performed, then we get

$$\frac{\partial^2 U_n}{\partial x_n^2} = - s_{44}(c_1 + c_2)\left(x_n^2 \frac{\partial^4 u_n}{\partial x_n^4} + 6x_n \frac{\partial^3 u_n}{\partial x_n^3} + \frac{\partial^2 u_n}{\partial x_n^2} \right), \tag{9.3}$$

If Equations (9.2), (9.3) are substituted to Equation (9.1), we get

$$x_n^2 \frac{\partial^4 u_n}{\partial x_n^4} + (7 - c_3)x_n \frac{\partial^3 u_n}{\partial x_n^3} + 4(2 - c_3)\frac{\partial^2 u_n}{\partial x_n^2} = 0\,, \tag{9.4}$$

If $u_n = x_n^{\lambda}$, then, with respect to Equation (9.4), we get

$$u_n = C_1 x_n + C_2 x_n^{c_3} + \frac{C_3}{x_n^2} + C_4\,, \tag{9.5}$$

where $c_3 < 0$ (see Equation (2.21)), and the integration constants C_1, C_2, C_3, C_4 are determined by the boundary conditions in Chapter 4. With regard to Equation (2.1)-(2.4), (2.17)-(2.20), (2.29), (9.5), we get

$$\varepsilon_n = C_1 + C_2 c_3 x_n^{c_3 - 1} - \frac{2 C_3}{x_n^3}, \tag{9.6}$$

$$\varepsilon_\varphi = \varepsilon_\nu = \frac{u_n}{x_n} = C_1 + C_2 x_n^{c_3 - 1} + \frac{C_3}{x_n^3} + \frac{C_4}{x_n}, \tag{9.7}$$

$$\varepsilon_{n\varphi} = s_{44} \sigma_{n\varphi} = \frac{\partial C_1}{\partial \varphi} + x_n^{c_3 - 1} \frac{\partial C_2}{\partial \varphi} + \frac{1}{x_n^3} \frac{\partial C_3}{\partial \varphi} + \frac{1}{x_n} \frac{\partial C_4}{\partial \varphi}, \tag{9.8}$$

$$\varepsilon_{n\theta} = s_{44} \sigma_{n\theta} = \Theta \left[\frac{\partial C_1}{\partial \nu} + x_n^{c_3 - 1} \frac{\partial C_2}{\partial \nu} + \frac{1}{x_n^3} \frac{\partial C_3}{\partial \nu} + \frac{1}{x_n} \frac{\partial C_4}{\partial \nu} \right], \tag{9.9}$$

$$\sigma_n = C_1 (c_1 - c_2) + C_2 [(c_1 + c_2) c_3 - 2c_2] x_n^{c_3 - 1} - \frac{2 C_3 (c_1 + 2c_2)}{x_n^3} - \frac{2 c_2 C_4}{x_n}, \tag{9.10}$$

$$\sigma_\varphi = \sigma_\nu = C_1 (c_1 - c_2) + C_2 (c_1 - c_2 c_3) x_n^{c_3 - 1} + \frac{C_3 (c_1 + 2c_2)}{x_n^3} + \frac{C_3 (c_1 + 2c_2)}{x_n^3} + \frac{c_1 C_4}{x_n}, \tag{9.11}$$

$$\sigma_1 = \eta_1 + \eta_2 x_n^{c_3 - 1} + \frac{\eta_3}{x_n^3} + \frac{\eta_4}{x_n}, \tag{9.12}$$

$$w = \kappa_1 + \kappa_2 x_n^{2(c_3 - 1)} + \frac{\kappa_3}{x_n^6} + \frac{\kappa_4}{x_n^2} + (\kappa_5 + \kappa_9) x_n^{c_3 - 1} + \frac{\kappa_6}{x_n^3} + \kappa_7 x_n^{c_3 - 4} + \frac{\kappa_8}{x_n} + \frac{\kappa_{10}}{x_n^4}, \tag{9.13}$$

where Θ and η_i, κ_i (i = 1,2,3) are given by Equations (1.15) and (8.13), respectively, and η_4, κ_j (j = 4,…,10) are derived as

$$\eta_4 = C_4\gamma_2 + \frac{1}{s_{44}}\left(\gamma_3\frac{\partial C_4}{\partial\varphi} + \gamma_4\frac{\partial C_4}{\partial\nu}\right),$$

$$\kappa_4 = c_1 C_4^2 + \frac{1}{s_{44}}\left[\left(\frac{\partial C_4}{\partial\varphi}\right)^2 + \Theta^2\left(\frac{\partial C_4}{\partial\nu}\right)^2\right],$$

$$\kappa_5 = (c_1 - c_2)(2 + c_3)C_1 C_2 + \frac{1}{s_{44}}\left(\frac{\partial C_1}{\partial\varphi}\frac{\partial C_2}{\partial\varphi} + \Theta^2\frac{\partial C_1}{\partial\nu}\frac{\partial C_2}{\partial\nu}\right),$$

$$\kappa_6 = \frac{2}{s_{44}}\left(\frac{\partial C_1}{\partial\varphi}\frac{\partial C_3}{\partial\varphi} + \Theta^2\frac{\partial C_1}{\partial\nu}\frac{\partial C_3}{\partial\nu}\right),$$

$$\kappa_7 = \left[2c_2(1 - c_3) - c_1\right]C_2 C_3 + \frac{2}{s_{44}}\left(\frac{\partial C_2}{\partial\varphi}\frac{\partial C_3}{\partial\varphi} + \Theta^2\frac{\partial C_2}{\partial\nu}\frac{\partial C_3}{\partial\nu}\right),$$

$$\kappa_8 = (c_1 - c_2)C_1 C_4 + \frac{1}{s_{44}}\left(\frac{\partial C_1}{\partial\varphi}\frac{\partial C_4}{\partial\varphi} + \Theta^2\frac{\partial C_1}{\partial\nu}\frac{\partial C_4}{\partial\nu}\right),$$

$$\kappa_9 = (c_1 - c_2 c_3)C_2 C_4 + \frac{1}{s_{44}}\left(\frac{\partial C_2}{\partial\varphi}\frac{\partial C_4}{\partial\varphi} + \Theta^2\frac{\partial C_2}{\partial\nu}\frac{\partial C_4}{\partial\nu}\right),$$

$$\kappa_{10} = (c_1 + 2c_2)C_3 C_4 + \frac{1}{s_{44}}\left(\frac{\partial C_3}{\partial\varphi}\frac{\partial C_4}{\partial\varphi} + \Theta^2\frac{\partial C_3}{\partial\nu}\frac{\partial C_4}{\partial\nu}\right). \qquad (9.14)$$

In case of the ellipsoidal inclusion, we get $C_{2IN} = C_{3IN} = C_{4IN} = 0$, otherwise $(u_{nIN})_{x_n=0} = \pm\infty$, $(\varepsilon_{nIN})_{x_n=0} = \pm\infty$, $(\sigma_{nIN})_{x_n=0} = \pm\infty$, due to $c_3 < 0$ (see Equation (2.21)). In case of $C_{1IN} \neq 0$, the mathematical solutions for the ellipsoidal inclusion is given by Equations (8.58)- (8.66).

9.2 Cell Matrix

The integration constants C_{1M}, C_{2M}, C_{3M}, C_{4M} are determined by Equations (4.1), (4.2) or (4.1)-(4.3), and then the following combinations are considered: $C_{1M} \neq 0$, $C_{4M} \neq 0$, $C_{2M} = C_{3M} = 0$; $C_{2M} \neq 0$, $C_{4M} \neq 0$, $C_{1M} = C_{3M} = 0$; $C_{3M} \neq 0$, $C_{4M} \neq 0$, $C_{1M} = C_{2M} = 0$; $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{4M} \neq 0$, $C_{3M} = 0$; $C_{1M} \neq 0$, $C_{3M} \neq 0$, $C_{4M} \neq 0$, $C_{2M} = 0$; $C_{2M} \neq 0$, $C_{3M} \neq 0$, $C_{4M} \neq 0$, $C_{1M} = 0$. The combinations of C_{1M}, C_{2M}, C_{3M} are presented in Chapter 8. Finally, such a combination is considered to exhibit mininum potential energy $W_p = W_{IN} + W_M$ (see Equation (2.30)).

Conditions $C_{1M} \neq 0$, $C_{4M} \neq 0$, $C_{2M} = C_{3M} = 0$. With regard to Equations (2.30), (4.1), (4.2), (9.5)-(9.13), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta}, \tag{9.15}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = \frac{u_{nM}}{x_n} = -\frac{p_n}{\zeta}\left(1-\frac{1}{x_n}\right), \tag{9.16}$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = -\left(1-\frac{1}{x_n}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right), \tag{9.15}$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = -\Theta\left(1-\frac{1}{x_n}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right), \tag{9.16}$$

$$\sigma_{nM} = -\frac{p_n}{\zeta}\left(c_{1M} - c_{2M} - \frac{2c_{2M}}{x_n}\right), \tag{9.17}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta}\left(c_{1M} - c_{2M} - \frac{c_{1M}}{x_n}\right), \tag{9.18}$$

$$\sigma_{1M} = \eta_{1M} + \frac{\eta_{4M}}{x_n}, \tag{9.19}$$

$$w_M = \kappa_{1M} + \frac{\kappa_{4M}}{x_n^2} + \frac{\kappa_{8M}}{x_n}, \tag{9.20}$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\kappa_{1M}}{3}\left(x_M^3 - x_{IN}^3\right) + \kappa_{4M}\left(x_M - x_{IN}\right)\right.$$

$$+\frac{\kappa_{8M}}{2}\left(x_M^2 - x_{IN}^2\right)\Bigg]\,\Omega\, d\varphi\, dv\,, \tag{9.21}$$

where Θ, Ω; x_{IN}, x_M; s_{44M}, c_{iM} (i = 1,2,3) are given by Equations (1.13); (1.15); (2.16), (2.21), respectively, and ς, η_{4M}, κ_{jM}, (i = 4,8; see Equation (9.14)) have the forms

$$\zeta = c_{1M} - c_{2M} + \frac{2c_{2M}\,x_M}{x_{IN}},$$

$$\eta_{4M} = \frac{p_n\gamma_{2M}}{\zeta} + \frac{1}{s_{44M}}\left[\gamma_{3M}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right) + \gamma_{3M}\frac{\partial}{\partial v}\left(\frac{p_n}{\zeta}\right)\right],$$

$$\kappa_{4M} = c_{1M}\left(\frac{p_n}{\zeta}\right)^2 + \frac{1}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)^2\right] + \Theta^2\left[\frac{\partial}{\partial v}\left(\frac{p_n}{\zeta}\right)^2\right]\right\},$$

$$\kappa_{8M} = \left(c_{1M} - c_{1M}\right)\left(\frac{p_n}{\zeta}\right)^2 - \frac{1}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)^2\right] + \Theta^2\left[\frac{\partial}{\partial v}\left(\frac{p_n}{\zeta}\right)^2\right]\right\}. \tag{9.22}$$

The coefficients η_{1M}, κ_{1M} are given by Equation (8.23), where ς in Equation (8.23) is given by Equation (9.22). The normal stress p_n is given by Equation (4.6). With regard to Equation (9.16), the coefficient ρ_M in Equation (4.6) is derived as

$$\rho_M = \frac{1}{\zeta}\left(1 - \frac{1}{x_{IN}}\right). \tag{9.23}$$

Conditions $C_{2M} \neq 0$, $C_{4M} \neq 0$, $C_{1M} = C_{3M} = 0$. With regard to Equations (2.30), (4.1), (4.2), (9.5)-(9.13), we get

$$\varepsilon_{nM} = -\frac{p_n c_{3M}}{\zeta}\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}, \tag{9.24}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = \frac{u_{nM}}{x_n} = -\frac{p_n}{\zeta}\left[\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} - \frac{1}{x_n}\right], \tag{9.25}$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = -\left[x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right) - \frac{1}{x_n}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right], \tag{9.26}$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = -\Theta\left[x_n^{c_{3M}-1}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right) - \frac{1}{x_n}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right], \tag{9.27}$$

$$\sigma_{nM} = -\frac{p_n}{\zeta}\left\{\left[\left(c_{1M} + 2c_{2M}\right)c_{3M} - 2c_{2M}\right]\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} + \frac{2c_{2M}}{x_n}\right\}, \tag{9.28}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta}\left[\left(c_{1M} - c_{2M}\,c_{3M}\right)\left(\frac{x_M}{x_n}\right)^{c_{3M}-1} - \frac{c_{1M}}{x_n}\right], \tag{9.29}$$

$$\sigma_{1M} = \eta_{2M}\, x_n^{c_{3M}-1} + \frac{\eta_{4M}}{x_n}, \tag{9.30}$$

$$w_M = \kappa_{2M}\, x_n^{2\left(c_{3M}-1\right)} + \frac{\kappa_{4M}}{x_n^2} + \kappa_{9M}\, x_n^{c_{3M}-1}, \tag{9.31}$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\kappa_{2M}}{2c_{3M}+1}\left(x_M^{2c_{3M}+1} - x_{IN}^{2c_{3M}+1}\right) + \frac{\kappa_{3M}}{3}\left(\frac{1}{x_{IN}^3} - \frac{1}{x_M^3}\right) + \frac{\kappa_{9M}}{c_{3M}+2}\left(x_M^{c_{3M}+2} - x_{IN}^{c_{3M}+2}\right)\right]\Omega\, d\varphi\, d\nu, \tag{9.32}$$

where Θ, Ω; x_{IN}, x_M; s_{44M}, c_{iM} (i = 1,2,3) are given by Equations (1.13); (1.15); (2.16), (2.21), respectively, and ς, κ_{2M} (Equation (8.13)), κ_{9M} (Equation (9.14)) have the forms

$$\zeta = \left[(c_{1M} + 2c_{2M})c_{3M} - 2c_{2M}\right]\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1} + \frac{2c_{2M}x_M}{x_M},$$

$$\kappa_{2M} = \left[\frac{c_{3M}^2\,(c_{1M} + 2c_{2M})}{2} + c_{1M} + 2c_{2M}\,c_{3M}\right]\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)^2$$
$$+ \frac{1}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\right]^2\right\},$$

$$\kappa_{9M} = \frac{c_{1M} - c_{2M}\,c_{3M}}{x_M^{c_{3M}-1}}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)^2$$
$$+ \frac{1}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right) + \Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\right].$$
(9.33)

The coefficients η_{2M} and η_{4M}, κ_{4M} are given by Equations (8.23) and (9.22), where ς in Equations (8.23), (9.22) is given by Equation (9.33). The normal stress p_n is given by Equation (4.6). With regard to Equation (9.25), the coefficient ρ_M in Equation (4.6) is derived as

$$\rho_M = \frac{1}{\zeta}\left[\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1} - \frac{1}{x_n}\right]. \tag{9.34}$$

Conditions $C_{3M} \neq 0$, $C_{4M} \neq 0$, $C_{1M} = C_{2M} = 0$. With regard to Equations (2.30), (4.1), (4.2), (9.5)-(9.13), we get

$$\varepsilon_{nM} = \frac{2\,p_n}{\zeta}\left(\frac{x_{IN}}{x_n}\right)^3, \tag{9.35}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = \frac{u_{nM}}{x_n} = -\frac{p_n}{\zeta}\left[\left(\frac{x_{IN}}{x_n}\right)^3 - \frac{1}{x_n}\right], \tag{9.36}$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = -\left[x_n^3 \frac{\partial}{\partial\varphi}\left(\frac{p_n x_{IN}^3}{\zeta}\right) - \frac{1}{x_n}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\right], \tag{9.37}$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = -\,\Theta^2\left[x_n^3 \frac{\partial}{\partial\nu}\left(\frac{p_n x_{IN}^3}{\zeta}\right) - \frac{1}{x_n}\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right], \tag{9.38}$$

$$\sigma_{nM} = -\frac{p_n}{\zeta}\left[\left(c_{1M} + 2c_{2M}\right)\left(\frac{x_{IN}}{x_n}\right)^3 - \frac{c_{2M}}{x_n}\right], \tag{9.39}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\frac{p_n}{\zeta}\left[\left(c_{1M} - 2c_{2M}\right)\left(\frac{x_{IN}}{x_n}\right)^3 - \frac{c_{1M}}{x_n}\right], \tag{9.40}$$

$$\sigma_{1M} = \frac{\eta_{3M}}{x_n^3} + \frac{\eta_{4M}}{x_n}, \tag{9.41}$$

$$w_M = \frac{\kappa_{3M}}{x_n^6} + \frac{\kappa_{4M}}{x_n^2} + \frac{\kappa_{10M}}{x_n^4}, \tag{9.42}$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\kappa_{3M}}{3}\left(\frac{1}{x_{IN}^3} - \frac{1}{x_M^3}\right) + \kappa_{4M}\left(x_M - x_{IN}\right) + \kappa_{10M}\left(\frac{1}{x_{IN}} - \frac{1}{x_M}\right)\right]\Omega\, d\varphi\, d\nu, \tag{9.43}$$

where Θ, Ω; x_{IN}, x_M; s_{44M}, c_{iM} (i = 1,2) are given by Equations (1.13), (1.15), (2.16), (2.21), respectively, and ς, κ_{4M} (see Equation (8.13)), κ_{10M} (see Equation (9.14)) have the forms

$$\zeta = \left[c_{3M}\left(c_{1M} + c_{2M}\right) - 2c_{2M}\right]\left(\frac{x_{IN}}{x_M}\right)^2,$$

$$\kappa_{10} = x_{IN}^3\left(c_{1M} + 2c_{2M}\right)\left(\frac{p_n}{\zeta}\right)^2 + \frac{1}{s_{44}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n x_{IN}}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right) + \Theta^2\frac{\partial}{\partial\nu}\left(\frac{p_n x_{IN}}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\right]. \tag{9.44}$$

The coefficients η_{3M}, κ_{3M} and η_{4M}, κ_{4M} are given by Equations (8.34) and (9.22), where ς in Equations (8.34), (9.22) is given by Equation (9.44). The normal stress p_n is given by Equation (4.6). With regard to Equation (8.37), the coefficient ρ_M in Equation (4.6) is derived as

$$\rho_M = \frac{x_{IN} - 1}{\zeta\, x_{IN}}. \tag{9.45}$$

Conditions $C_{1M} \neq 0$, $C_{2M} \neq 0$, $C_{4M} \neq 0$, $C_{3M} = 0$. With regard to Equa-tions (2.30), (4.1)-(4.3), (9.5)-(9.13), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta}\left[1 - \left(\frac{x_{IN}}{x_n}\right)^{c_{3M}-1}\right], \tag{9.46}$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = \frac{u_{nM}}{x_n} = -\frac{p_n}{\zeta}\left\{1 - \frac{1}{c_{3M}}\left[\left(\frac{x_n}{x_M}\right)^{c_{3M}-1} - \frac{\left(c_{3M}-1\right)x_M}{x_n}\right]\right\} \tag{9.47}$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = -\left\{\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right) - \frac{1}{c_{3M}}\left[x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\right] + \frac{c_{3M}-1}{c_{3M}\, x_n}\frac{\partial}{\partial\varphi}\left(\frac{p_n x_M}{\zeta}\right)\right\}, \tag{9.48}$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M}$$

$$= -\,\Theta^2 \left\{ \frac{\partial}{\partial \varphi}\left(\frac{p_n}{\zeta}\right) - \frac{1}{c_{3M}}\left[x_n^{c_{3M}-1} \frac{\partial}{\partial \varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}} \right) \right] \right.$$

$$- \frac{1}{c_{3M}}\left[x_n^{c_{3M}-1} \frac{\partial}{\partial \varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}} \right) \right]$$

$$\left. + \frac{c_{3M}-1}{c_{3M}\, x_n} \frac{\partial}{\partial \varphi}\left(\frac{p_n\, x_M}{\zeta} \right) \right\}, \tag{9.49}$$

$$\sigma_{nM} = -\,\frac{p_n}{\zeta}\left[c_{1M} - c_{2M} - \frac{(c_{1M}+c_{2M})c_{3M} - 2c_{2M}}{c_{3M}} \left(\frac{x_n}{x_M}\right)^{c_{3M}-1} \right.$$

$$\left. + \frac{2c_{2M}(c_{3M}-1)x_M}{c_{3M}\, x_n} \right], \tag{9.50}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\,\frac{p_n}{\zeta}\left[c_{1M} - c_{2M} - \frac{c_{1M} + c_{2M}\, c_{3M}}{c_{3M}} \left(\frac{x_n}{x_M}\right)^{c_{3M}-1} \right.$$

$$\left. + \frac{c_{1M}(c_{3M}-1)x_M}{c_{3M}\, x_n} \right], \tag{9.51}$$

$$\sigma_{1M} = \eta_{1M} + \eta_{2M}\, x_n^{c_{3M}-1} + \frac{\eta_{4M}}{x_n}, \tag{9.52}$$

$$w_M = \kappa_{1M} + \kappa_{2M}\, x_n^{2(c_{3M}-1)} + \frac{\kappa_{4M}}{x_n^2} + (\kappa_{5M}+\kappa_{9M})x_n^{c_{3M}-1} + \frac{\kappa_{8M}}{x_n}, \tag{9.53}$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2} \left[\frac{\kappa_{1M}}{3}\left(x_M^3 - x_{IN}^3\right) + \frac{\kappa_{2M}}{2c_{3M}+1}\left(x_M^{2c_{3M}+1} - x_{IN}^{2c_{3M}+1}\right) \right.$$

$$+ \kappa_{4M}(x_M - x_{IN}) + \frac{\kappa_{5M}+\kappa_{9M}}{c_{3M}+2}\left(x_M^{c_{3M}+2} - x_{IN}^{c_{3M}+2}\right)$$

$$+\frac{\kappa_{8M}}{2}\left(x_M^2 - x_{IN}^2\right)\bigg]\,\Omega\, d\varphi\, d\nu\,, \tag{9.54}$$

where Θ, Ω; x_{IN}, x_{M}; $s_{44\mathrm{M}}$, $c_{i\mathrm{M}}$ (i = 1,2,3) are given by Equations (1.13), (1.15), (2.16), (2.21), respectively, and ς, $\kappa_{j\mathrm{M}}$ (j = 4,5,8,9; see Equation (9.14)) have the forms

$$\zeta = c_{1M} - c_{2M} - \frac{\left(c_{1M} + c_{2M}\right)c_{3M} - 2c_{2M}}{c_{3M}}\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1}$$

$$+\frac{2c_{2M}\left(c_{3M}-1\right)x_M}{c_{3M}\,x_{IN}}\,,$$

$$\kappa_{4M} = c_{1M}\left[\frac{\left(c_{3M}-1\right)x_M\,p_n}{\zeta\,c_{3M}}\right]^2$$

$$+\frac{1}{s_{44M}}\left(\frac{c_{3M}-1}{c_{3M}}\right)^2\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\,x_M}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\,x_M}{\zeta}\right)\right]^2\right\},$$

$$\kappa_{5M} = -\frac{\left(c_{1M}-c_{2M}\right)\left(2+c_{3M}\right)}{c_{3M}\,x_M^{c_{3M}-1}}\left(\frac{p_n}{\zeta}\right)^2$$

$$-\frac{2}{s_{44M}\,c_{3M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\,x_M^{c_{3M}-1}}\right)\right.$$

$$\left.+\Theta^2\,\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\,x_M^{c_{3M}-1}}\right)\right],$$

$$\kappa_{8M} = -\frac{\left(c_{1M}-c_{2M}\right)\left(c_{3M}-1\right)x_M}{c_{3M}}\left(\frac{p_n}{\zeta}\right)^2$$

$$-\frac{c_{3M}-1}{s_{44M}\,c_{3M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\,x_M}{\zeta}\right)\right.$$

$$\left.+\Theta^2\,\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\,x_M}{\zeta}\right)\right],$$

$$\kappa_{9M} = \frac{(c_{1M}-c_{2M}\,c_{3M})(c_{3M}-1)}{x_M^{c_{3M}-2}}\left(\frac{p_n}{\zeta\,c_{3M}}\right)^2$$

$$+\frac{(c_{3M}-1)}{s_{44M}\,c_{3M}^2}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\,x_M^{c_{3M}-1}}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\,x_M}{\zeta}\right)\right.$$

$$\left.+\,\Theta^2\,\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\,x_M^{c_{3M}-1}}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\,x_M}{\zeta}\right)\right]. \qquad (9.55)$$

The coefficients η_{iM}, κ_{iM} (i = 1,2) and η_{4M} are given by Equations (8.23) and (9.22), where ς in Equations (8.23), (9.22) is given by Equation (9.55). The normal stress p_n is given by Equation (4.6). With regard to Equation (8.37), the coefficient ρ_M in Equation (4.6) is derived as

$$\rho_M = \frac{1}{\zeta}\left\{1-\frac{1}{c_{3M}}\left[\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1} - \frac{(c_{3M}-1)x_M}{x_{IN}}\right]\right\}. \qquad (9.56)$$

Conditions $C_{1M} \neq 0$, $C_{3M} \neq 0$, $C_{4M} \neq 0$, $C_{2M} = 0$. With regard to Equa-tions (2.30), (4.1)-(4.3), (9.5)-(9.13), we get

$$\varepsilon_{nM} = -\frac{p_n}{\zeta}\left[1-\frac{3}{2}\left(\frac{x_M}{x_n}\right)^3\right], \qquad (9.57)$$

$$\varepsilon_{\varphi M} = \varepsilon_{\theta M} = \frac{u_{nM}}{x_n} = -\frac{p_n}{\zeta}\left[1+\frac{1}{2}\left(\frac{x_M}{x_n}\right)^3 - \frac{3\,x_M}{2\,x_n}\right] \qquad (9.58)$$

$$\varepsilon_{n\varphi M} = s_{44M}\,\sigma_{n\varphi M} = -\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)+\frac{1}{2x_n^3}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,x_M^3}{\zeta}\right)\right.$$

$$\left.-\frac{3}{2\,x_n}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,x_M}{\zeta}\right)\right], \qquad (9.59)$$

$$\varepsilon_{n\theta M} = s_{44M}\,\sigma_{n\theta M} = -\,\Theta^2\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right) + \frac{1}{2x_n^3}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,x_M^3}{\zeta}\right) - \frac{3}{2x_n}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,x_M}{\zeta}\right)\right], \quad (9.60)$$

$$\sigma_{nM} = -\,\frac{p_n}{\zeta}\left[c_{1M} - c_{2M} - \left(c_{1M} + 2c_{2M}\right)\left(\frac{x_M}{x_n}\right)^3 + \frac{3c_{2M}\,x_M}{x_n}\right], \quad (9.61)$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = -\,\frac{p_n}{\zeta}\left[c_{1M} - c_{2M} + \frac{c_{1M} + 2c_{2M}}{2}\left(\frac{x_M}{x_n}\right)^3 - \frac{3c_{1M}\,x_M}{x_n}\right] \quad (9.62)$$

$$\sigma_{1M} = \eta_{1M} + \frac{\eta_{3M}}{x_n^3} + \frac{\eta_{4M}}{x_n}, \quad (9.63)$$

$$w_M = \kappa_{1M} + \frac{\kappa_{3M}}{x_n^6} + \frac{\kappa_{4M}}{x_n^2} + \frac{\kappa_{6M}}{x_n^3} + \frac{\kappa_{8M}}{x_n} + \frac{\kappa_{10M}}{x_n^4}, \quad (9.64)$$

$$W_M = 4\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\kappa_{1M}}{3}\left(x_M^3 - x_{IN}^3\right) + \frac{\kappa_{3M}}{3}\left(\frac{1}{x_{IN}^3} - \frac{1}{x_M^3}\right) + \kappa_{4M}\left(x_M - x_{IN}\right) + \kappa_{6M}\ln\left(\frac{x_M}{x_{IN}}\right) + \frac{\kappa_{8M}}{2}\left(x_M^2 - x_{IN}^2\right) + \kappa_{10M}\left(\frac{1}{x_{IN}} - \frac{1}{x_{IM}}\right)\right]\Omega\,d\varphi\,d\nu, \quad (9.65)$$

where Θ, Ω; x_{IN}, x_M; s_{44M}, c_{iM} ($i = 1,2,3$) are given by Equations (1.13), (1.15), (2.16), (2.21), respectively, and ς, κ_{3M} (see Equation (8.13)), κ_{iM} ($j = 4,6,8,10$; see Equation (9.14)) have the forms

$$\zeta = c_{1M} - c_{2M} - \left(c_{1M} + 2c_{2M}\right)\left(\frac{x_M}{x_{IN}}\right)^3 + \frac{3c_{2M}\,x_M}{x_{IN}}\,,$$

$$\kappa_{3M} = 3\left(c_{1M}+2c_{2M}\right)\left(\frac{p_n\,x_M^3}{2\zeta}\right)^2$$

$$+\frac{1}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\,x_M^3}{2\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\,x_M^3}{2\zeta}\right)\right]^2\right\},$$

$$\kappa_{4M} = c_{1M}\left(\frac{3\,p_n\,x_M}{2\zeta}\right)^2$$

$$+\frac{1}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{3\,p_n\,x_M}{2\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{3\,p_n\,x_M}{2\zeta}\right)\right]^2\right\},$$

$$\kappa_{6M} = \frac{2}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\,x_M^3}{2\,\zeta}\right)\right.$$

$$\left.+\Theta^2\,\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\,x_M^3}{2\zeta}\right)\right],$$

$$\kappa_{8M} = -\,\frac{3x_M\left(c_{1M}-c_{2M}\right)}{2}\left(\frac{p_n}{\zeta}\right)^2$$

$$-\,\frac{3}{2s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\,x_M}{\zeta}\right)\right.$$

$$\left.+\Theta^2\,\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\,x_M}{\zeta}\right)\right],$$

$$\kappa_{10M} = -\,\frac{3\left(c_{1M}+2c_{3M}\right)}{4}\left(\frac{p_n\,x_M^2}{\zeta}\right)^2$$

$$-\,\frac{3}{4s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\,x_M^3}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\,x_M}{\zeta}\right)\right.$$

$$+\,\Theta^2\,\frac{\partial}{\partial\nu}\left(\frac{p_n\,x_M^3}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\,x_M}{\zeta}\right)\Bigg]. \tag{9.66}$$

The coefficients η_{iM} (i = 1,3), κ_{1M} and η_{4M} are given by Equations (8.23) and (9.22), where ς in Equations (8.23), (9.22) is given by Equation (9.66). The normal stress p_n is given by Equation (4.6). With regard to Equation (9.58), the coefficient ρ_M in Equation (4.6) is derived as

$$\rho_M=\frac{1}{\zeta}\left[1+\frac{1}{2}\left(\frac{x_M}{x_{IN}}\right)^3-\frac{3x_M}{2x_{IN}}\right]. \tag{9.67}$$

Conditions $C_{2M}\neq 0$, $C_{3M}\neq 0$, $C_{4M}\neq 0$, $C_{1M}=0$. With regard to Equa-tions (2.30), (4.1)-(4.3), (9.5)-(9.13), we get

$$\varepsilon_{nM}=-\frac{c_{3M}\,p_n}{\zeta}\left[\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}-\left(\frac{x_M}{x_n}\right)^3\right], \tag{9.68}$$

$$\varepsilon_{\varphi M}=\varepsilon_{\theta M}=\frac{u_{nM}}{x_n}$$
$$=-\frac{p_n}{\zeta}\left[\left(\frac{x_n}{x_M}\right)^{c_{3M}-1}+\frac{c_{3M}}{2}\left(\frac{x_M}{x_n}\right)^3-\frac{(c_{3M}+2)x_M}{2x_n}\right], \tag{9.69}$$

$$\varepsilon_{n\varphi M}=s_{44M}\,\sigma_{n\varphi M}$$
$$=-\left[x_n^{c_{3M}-1}\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\,x_M^{c_{3M}-1}}\right)+\frac{c_{3M}}{2x_n^3}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,x_M^3}{\zeta}\right)\right.$$
$$\left.-\frac{c_{3M}+2}{2x_n}\frac{\partial}{\partial\varphi}\left(\frac{p_n\,x_M}{\zeta}\right)\right], \tag{9.70}$$

$$\varepsilon_{n\theta M}=s_{44M}\,\sigma_{n\theta M}$$

$$= -\Theta^2 \left[x_n^{c_{3M}-1} \frac{\partial}{\partial \nu} \left(\frac{p_n}{\zeta \, x_M^{c_{3M}-1}} \right) + \frac{c_{3M}}{2\, x_n^3} \frac{\partial}{\partial \nu} \left(\frac{p_n \, x_M^3}{\zeta} \right) \right.$$

$$\left. - \frac{c_{3M}+2}{2\, x_n} \frac{\partial}{\partial \nu} \left(\frac{p_n \, x_M}{\zeta} \right) \right], \tag{9.71}$$

$$\sigma_{nM} = - \frac{p_n}{\zeta} \left\{ \left[\left(c_{1M} + c_{2M} \right) c_{3M} - 2\, c_{2M} \right] \left(\frac{x_n}{x_M} \right)^{c_{3M}-1} \right.$$

$$\left. - \left(c_{1M} \, c_{3M} + 2\, c_{2M} \right) \left(\frac{x_M}{x_n} \right)^3 + \frac{c_{2M} \left(c_{3M} + 2 \right) x_M}{x_n} \right\}, \tag{9.72}$$

$$\sigma_{\varphi M} = \sigma_{\theta M} = - \frac{p_n}{\zeta} \left[\left(c_{1M} + c_{2M} \, c_{3M} \right) \left(\frac{x_n}{x_M} \right)^{c_{3M}-1} \right.$$

$$\left. + \frac{c_{3M} \left(c_{1M} + 2\, c_{2M} \right)}{2} \left(\frac{x_M}{x_n} \right)^3 - \frac{c_{1M} \left(c_{3M} + 2 \right) x_M}{2\, x_n} \right\}, \tag{9.73}$$

$$\sigma_{1M} = \eta_{2M} \, x_n^{c_{3M}-1} + \frac{\eta_{3M}}{x_n^3} + \frac{\eta_{4M}}{x_n}, \tag{9.74}$$

$$w_M = \kappa_{2M} \, x_n^{2\left(c_{3M}-1 \right)} + \frac{\kappa_{3M}}{x_n^6} + \frac{\kappa_{4M}}{x_n^2} + \kappa_{7M} \, x_n^{c_{3M}-4}$$

$$+ \kappa_{9M} \, x_n^{c_{3M}-1} + \frac{\kappa_{10M}}{x_n^4}, \tag{9.75}$$

$$W_M = 4 \int_0^{\pi/2} \int_0^{\pi/2} \left[\frac{\kappa_{2M}}{2\, c_{3M}+1} \left(x_M^{2\, c_{3M}+1} - x_{IN}^{2\, c_{3M}+1} \right) + \frac{\kappa_{3M}}{3} \left(\frac{1}{x_{IN}^3} - \frac{1}{x_M^3} \right) \right.$$

$$+\kappa_{4M}\left(x_M - x_{IN}\right) + \frac{\kappa_{7M}}{c_{3M}-1}\left(x_M^{c_{3M}-1} - x_{IN}^{c_{3M}-1}\right)$$

$$+\frac{\kappa_{9M}}{c_{3M}+2}\left(x_M^{c_{3M}+2} - x_{IN}^{c_{3M}+2}\right)$$

$$+\kappa_{10M}\left(\frac{1}{x_{IN}} - \frac{1}{x_{IM}}\right)\Bigg]\Omega\, d\varphi\, d\nu\,, \qquad (9.76)$$

where Θ, Ω; x_{IN}, x_M; s_{44M}, c_{iM} (i = 1,2,3) are given by Equations (1.13), (1.15), (2.16), (2.21), respectively, and ς, κ_{iM} (i = 2,3; see Equation (8.13)), κ_{iM} (j = 4,7,9,10; see Equation (9.14)) have the forms

$$\zeta = \Bigg\{ \left[\left(c_{1M} + 2c_{2M}\right)c_{3M} - 2c_{2M}\right]\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1} + \frac{3c_{2M}\, x_M}{x_{IN}}$$

$$- c_{3M}\left(c_{1M} + 2c_{2M}\right)\left(\frac{x_M}{x_{IN}}\right)^3 + \frac{c_{3M}\left(c_{3M}+2\right)x_M}{x_{IN}}\Bigg\},$$

$$\kappa_{2M} = \left[\frac{\left(c_{1M}+c_{2M}\right)c_{3M}^2}{2} + c_{1M} - 2c_{2M}\, c_{3M}\right]\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)^2$$

$$+\frac{1}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\right]^2\right\},$$

$$\kappa_{3M} = 3\left(c_{1M} + 2c_{2M}\right)\left(\frac{p_n\, c_{3M}\, x_M^3}{2\zeta}\right)^2$$

$$+\frac{c_{3M}^2}{2s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\, x_M}{\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\, x_M}{\zeta}\right)\right]^2\right\},$$

$$\kappa_{4M} = c_{1M}\left[\frac{p_n\, x_M\left(c_{3M}+2\right)}{2\zeta}\right]^2$$

$$+\frac{c_{3M}+2}{s_{44M}}\left\{\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\, x_M}{2\zeta}\right)\right]^2 + \Theta^2\left[\frac{\partial}{\partial\nu}\left(\frac{p_n\, x_M}{2\zeta}\right)\right]^2\right\},$$

$$\kappa_{7M} = \frac{c_{3M}\left[2\left(1-c_{3M}\right)c_{2M}-c_{1M}\right]}{2\,x_{3M}^{c_{3M}-4}}\left(\frac{p_n}{\zeta}\right)^2 + \frac{c_{3M}}{s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\, x_M^3}{\zeta}\right) + \Theta^2\,\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\, x_M^3}{\zeta}\right)\right],$$

$$\kappa_{9M} = -\frac{x_M\left(c_{1M}-c_{2M}\,c_{3M}\right)\left(c_{3M}+2\right)}{2\,x_M^{c_{3M}-1}}\left(\frac{p_n}{\zeta}\right)^2 - \frac{c_{3M}+2}{2\,s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\, x_M}{\zeta}\right) + \Theta^2\,\frac{\partial}{\partial\nu}\left(\frac{p_n}{\zeta\, x_M^{c_{3M}-1}}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\, x_M}{\zeta}\right)\right],$$

$$\kappa_{10M} = -c_{3M}\left(c_{1M}+2c_{2M}\right)\left(c_{3M}+2\right)\left(\frac{p_n\, x_M^2}{2\zeta}\right)^2 - \frac{c_{3M}\left(c_{3M}+2\right)}{4\,s_{44M}}\left[\frac{\partial}{\partial\varphi}\left(\frac{p_n\, x_M^3}{\zeta}\right)\frac{\partial}{\partial\varphi}\left(\frac{p_n\, x_M}{\zeta}\right) + \Theta^2\,\frac{\partial}{\partial\nu}\left(\frac{p_n\, x_M^3}{\zeta}\right)\frac{\partial}{\partial\nu}\left(\frac{p_n\, x_M}{\zeta}\right)\right]. \qquad (9.77)$$

The coefficients η_{iM} (i = 2,3) and η_{4M} are given by Equations (8.23) and (9.22), where ς in Equations (8.23), (9.22) is given by Equation (9.77). The normal stress p_n is given by Equation (4.6). With regard to Equation (9.58), the coefficient ρ_M in Equation (4.6) is derived as

$$\rho_M = \frac{1}{\zeta}\left[\left(\frac{x_{IN}}{x_M}\right)^{c_{3M}-1} + \frac{c_{3M}}{2}\left(\frac{x_M}{x_{IN}}\right)^3 - \frac{\left(c_{3M}+2\right)x_M}{2\,x_{IN}}\right]. \qquad (9.78)$$

CHAPTER 10

MATERIAL STRENGTHENING

The mathematical model of the material micro-strengthening $\sigma_{st}=\sigma_{st}(x_1)$ and the material macro-strengthening $\overline{\sigma_{st}}$ results from the following analysis (Ceniga 2008, 102-105). Figures 10.1 and 10.2 show the plane $x'_2x'_3$ in the cubic cell (see Figure 1.2) for $x_1\in\langle 0, a_1\rangle$ c, respectively, where $[x_1,x_2,x_3]$ are coordinates of the point $P \subset x'_2x'_3$, and a_1 is a dimension of the ellipsoidal inclusion along the axis x_1 (see Figure 1.2).

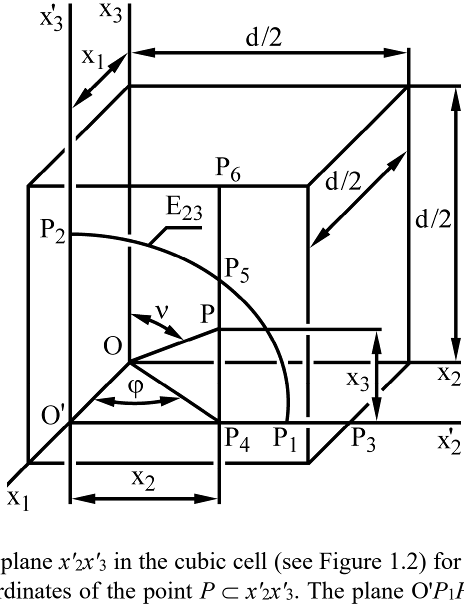


Figure 10.1. The plane $x'_2x'_3$ in the cubic cell (see Figure 1.2) for $x_1\in\langle 0, a_1\rangle$, where $[x_1,x_2,x_3]$ are coordinates of the point $P \subset x'_2x'_3$. The plane $O'P_1P_2$ with the ellipse E_{23} represents a cross section of the ellipsoid inclusion in the plane $x'_2x'_3$.

The plane $O'P_1P_2$ with the ellipse E_{23} (see Figure 10.1) represents a cross section of the ellipsoid inclusion in the plane $x'_2x'_3$. With regard to Figures

8.1, 8.2, the goniometric functions in Equations (1.7)-(1.16) have the forms

$$\sin\varphi = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}, \quad \cos\varphi = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \quad \tan\varphi = \frac{x_2}{x_1},$$

$$\sin\nu = \sqrt{\frac{x_1^2 + x_2^2}{x_1^2 + x_2^2 + x_3^2}}, \quad \cos\nu = \frac{x_{31}}{\sqrt{x_1^2 + x_2^2 + x_3^2}}, \quad x_n = \frac{x_3}{\cos\theta}, \qquad (10.1)$$

where $\cos\theta$ is given by Equation (1.12).

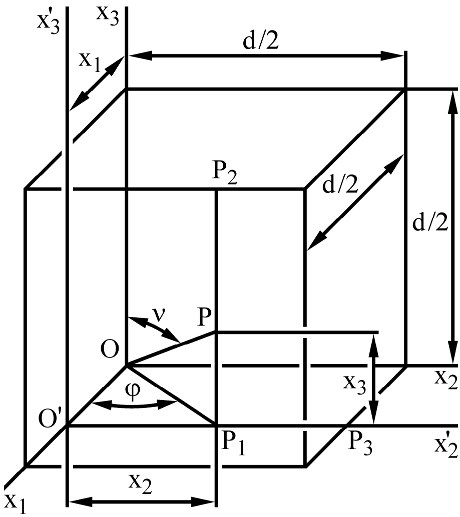


Figure 10.2. The plane $x'_2x'_3$ in the cubic cell (see Figure 1.2) for $x_1 \in \langle a_1, d/2\rangle$, where $[x_1, x_2, x_3]$ are coordinates of the point $P \subset x'_2x'_3$.

With regard to Equation (1.2), the parameters b_2, b_3 of the ellipse E_{23} along the axes x'_2, x'_3, respectively, are derived as (see Figure 8.1)

$$b_2 = O'P_1 = \frac{a_2\sqrt{a_1^2 - x_1^2}}{a_1}, \quad b_3 = O'P_2 = \frac{a_3\sqrt{a_1^2 - x_1^2}}{a_1}, \qquad (10.2)$$

and then we get

$$b_4 = P_4P_5 = \frac{a_3\sqrt{b_2^2 - x_2^2}}{a_2}. \tag{10.3}$$

The material micro-strengthening $\sigma_{st} = \sigma_{st}(x_1)$ represents a stress along the axis x_1, which is homogeneous at each point of the plane $x'_2x'_3$ with the area $S = d^2/4$, i.e., $\sigma_{st} \neq f(x_2, x_3)$.

If $x_1 \in \langle 0, a_1 \rangle$, then the elastic energy surface density W_{st}, which is induced by σ_{st}, and accumulated within the area $S_{IN} = \pi b_2 b_3 /4$ of the plane O'P_1P_2 and within the area $S_M = (d/2)^2 - S_{IN}$ of the plane $x'_2x'_3$ (see Figure 10.1), has the form

$$W_{st} = \omega \sigma_{1st}^2, \tag{10.4}$$

where σ_{1st} is related to the interval $x_1 \in \langle 0, a_1 \rangle$, and the coefficient ω is derived as

$$\omega = \frac{1}{8}\left[\pi b_2 b_3\left(\frac{1}{E_{IN}} - \frac{1}{E_M}\right) + \frac{d^2}{E_M}\right], \tag{10.5}$$

where E_{IN} and E_M is Young's modulus for the ellipsoidal inclusion and the matrix, respectively. The elastic energy surface density W_{1S}, which is induced by the stresses $\sigma_{1q} = \sigma_{1q}(x_1)$ ($q = IN,M$; see Equations (5.33), (8.62), (6.22), (6.33), (6.44), (6.55), (7.21), (7.32), (7.43), (7.54), (8.20), (8.31), (8.42), (8.53), (9.19), (9.30), (9.41), (9.52), (9.63), (9.74)), has the form

$$W_{1S} = \frac{1}{2}\left(\frac{W_{INS}}{E_{IN}} + \frac{W_{1MS}}{E_M}\right),$$

$$W_{INS} = \int_0^{b_2}\left(\int_0^{b_4} \sigma_{1IN}^2 \, dx_3\right) dx_2,$$

$$W_{1MS} = \int_0^{b_2}\left(\int_{b_4}^{d/2} \sigma_{1M}^2 \, dx_3\right) dx_2 + \int_0^{d/2}\left(\int_{b_4}^{d/2} \sigma_{1M}^2 \, dx_3\right) dx_2, \quad x_1 \in \langle 0, a_1 \rangle. \tag{10.6}$$

The material micro-strengthening $\sigma_{1st} = \sigma_{1st}(x_1)$ for $x_1 \in \langle a_1, d/2\rangle$, which results from the condition $W_{st} = W_{1S}$ (Ceniga 2008, 102-105), is derived as

$$\sigma_{1st} = \sqrt{\frac{1}{2\omega}\left(\frac{W_{INS}}{E_{IN}} + \frac{W_{1MS}}{E_M}\right)}, \quad x_1 \in \langle 0, a_1 \rangle . \tag{10.7}$$

If $x_1 \in \langle 0, a_1 \rangle$, then the elastic energy surface density W_{st}, which is induced by σ_{st} and accumulated within the area $S_M = d^2/4$ of the plane $x'_2 x'_3$ (see Figure 10.2), has the form

$$W_{st} = \frac{\sigma_{2st}^2 d^2}{8 E_M}, \tag{10.8}$$

where σ_{2st} is related to the interval $x_1 \in \langle a_1, d/2\rangle$. Similarly, we get

$$W_{2S} = \frac{W_{2MS}}{2 E_M}, \quad W_{2MS} = \int_0^{d/2} \int_{b_4}^{d/2} \sigma_1^2 \, dx_2 \, dx_3, \quad x_1 \in \left\langle a_1, \frac{d}{2} \right\rangle . \tag{10.9}$$

With regard to the condition $W_{st} = W_{2S}$ (Ceniga 2008, 102-105), we get

$$\sigma_{2st} = \frac{2\sqrt{W_{2S}}}{d} . \tag{10.10}$$

Finally, the material macro-strengthening $\overline{\sigma_{st}}$ is derived as (Ceniga 2008, 102-105)

$$\overline{\sigma_{st}} = \frac{2}{d}\left(\int_0^{a_1} \sigma_{1st} \, dx_1 + \int_0^{d/2} \sigma_{2st} \, dx_1 \right). \tag{10.11}$$

If $\beta_{IN} < \beta_M$ or $\beta_{IN} > \beta_M$, the material strengthening exhibits a resistive effect against compressive or tensile mechanical loading, respectively, where β_q ($q = M,IN$) is given by Equations (3.1)-(3.6), (3.17), (3.18).

The material macro-strengthening $\overline{\sigma_{st}} = \overline{\sigma_{st}}\left(v_{IN}, a_1, a_2, a_3\right)$ is a func-tion of the inclusion volume fraction v_{IN} and the dimensions a_1, a_2, a_3 of the ellip-soidal inclusion (see Equation (1.1)). In case of a real matrix-inclusion composite, such values of the microstructural parameters v_{IN}, a_1, a_2, a_3 can be numerically determined to result in a maximum value of $\left|\overline{\sigma_{st}}\right|$.

CHAPTER 11

MATERIAL CRACK FORMATION

11.1 Mathematical Procedure

The mathematical model of the crack formation in the cell matrix and the ellipsoidal inclusion results from the following analysis. Figure 11.1a and 11.1b shows a solid continuum with the volume V in the Cartesian system ($Ox_1x_2x_3$) without and with a crack, respectively.

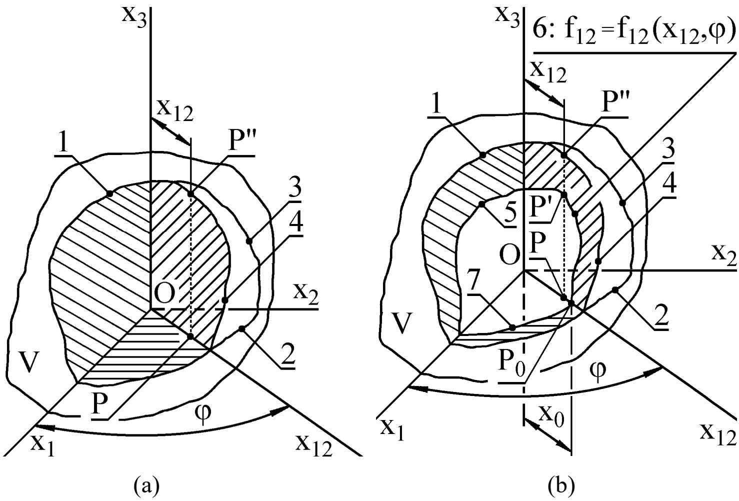


Figure 11.1. The solid continuum of a general shape with the volume V in the Cartesian system ($Ox_1x_2x_3$) (a) without and (b) with a crack, respectively. The crack is formed in the plane x_1x_2. The shaded area repre-sents cuts of the solid continuum in x_1x_3, x_1x_2, $x_{12}x_3$, where $x_{12} \subset x_1x_2$, $\varphi = \angle\,(x_1,x_{12}) \in \langle 0,\pi/2\rangle$, $x_{12}x_3 \perp x_1x_2$. The curves *1*, *2*, *3*, *4* are outlines of the cuts in the planes x_1x_3, x_1x_2, x_2x_3, $x_{12}x_3$, respectively. P and P'' are points of the axis x_{12} and the curve *4*, respectively. The curves *5* and *6* represent the crack shape in x_1x_3, $x_{12}x_3$, respectively. The crack shape in $x_{12}x_3$ is defined by the function $f_{12} = f_{12}\,(\varphi, x_{12}, x_3)$, which is determined by the cylindrical coordinates (φ, x_{12},x_3). The curve *7* defines a position of the crack tip in $x_{12}x_3$. $P_0 \subset x_1x_2$ represents the crack tip related to the plane $x_{12}x_3$.

The crack is formed in the plane x_1x_2. The shaded area represents cuts of the solid continuum in x_1x_3, x_1x_2, $x_{12}x_3$, where $x_{12} \subset x_1x_2$, $\varphi = \angle\, (x_1, x_{12}) \in \langle 0, \pi/2 \rangle$, $x_{12}x_3 \perp x_1x_2$. The curves *1*, *2*, *3*, *4* are outlines of the cuts in the planes x_1x_3, x_1x_2, x_2x_3, $x_{12}x_3$, respectively. Additionally, P and P'' represent points of the axis x_{12} and the curve *4*, respectively. The curves *5* and *6* represent the crack shape in x_1x_3, $x_{12}x_3$, respectively. The crack shape in $x_{12}x_3$ is defined by the function $f_{12} = f_{12}\,(x_{12}, \varphi,)$, which is determined by the cylindrical coordinates (x_{12}, x_3) for the parameter $\varphi \in \langle 0, \pi/2 \rangle$. The curve *7* defines a position of the crack tip in $x_{12}x_3$. The point $P_0 \subset x_1x_2$ represents the crack tip related to the plane $x_{12}x_3$.

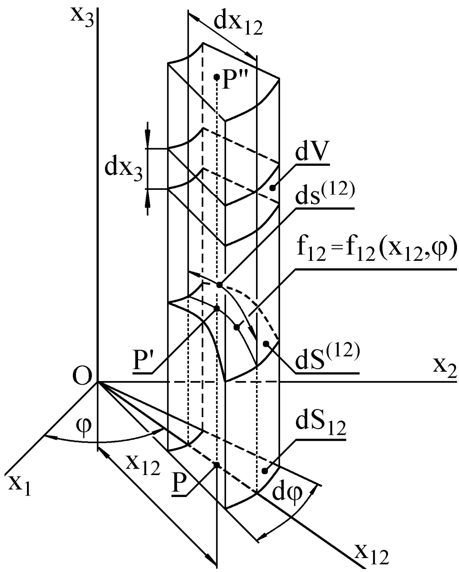


Figure 11.2. The infinitesimal prism with the height |PP''| (see Figure 11.1b) and the infinitesimal surface $dS_{12} = x_{12}\, d\varphi\, dx_{12}$ in x_1x_2. The infinitesimal crack surface $dS^{(12)}$ is related to the infinitesimal crack length $ds^{(12)}$ at the point P' (see Figure 11.1b). The function $f_{12} = f_{12}(x_{12}, \varphi)$ of the variable x_{12} for the parameter $\varphi \in \langle 0, \pi/2 \rangle$ defines the crack shape in the plane $x_{12}x_3$.

Let the energy $W = \int_V w dV$ be accumulated in the volume V, where w is energy density. Let W tend to be released by the crack formation in $x_{12}x_3$. The same is also valid for the infinitesimal energy dW, accumulated in the volume $dV = \int_{P}^{P''} dS_{12}\, dx_3$ of the infinitesimal prism (see Figure 11.2), where $dS_{12} = x_{12}\, d\varphi\, dx_3$, and then we get

$$dW = \int_{P}^{P''} w\, dV = \int_{P}^{P''} w\, dS_{12}\, dx_3 = W_c^{(12)}\, x_{12}\, d\varphi\, dx_{12} \quad , \tag{11.1}$$

where the curve integral $W_c^{(12)}$ has the form

$$W_c^{(12)} = \int_{P}^{P''} w\left(x_{12}, \varphi, x_3\right) dx_3 \; . \tag{11.2}$$

The crack is formed in the plane $x_1\, x_2$ provided that the condition

$$\int_{P}^{P''} w\left(x_{12}, \varphi, x_3\right) dx_3 = \int_{P}^{P''(-)} w\left(x_{12}, \varphi, -x_3\right) dx_3, \quad \varphi \in \left\langle 0, 2\pi \right\rangle \tag{11.3}$$

is valid for $\varphi \in \langle 0, \pi/2 \rangle$, where $P''^{(-)}$ is an intersection point of the line PP'' with the solid continuum surface for $x_3 < 0$. The energy dW is in an equilibrium state with the energy $dW_{cs} = \gamma\, dS^{(12)}$, which creates the infinitesimal crack surface $dS^{(12)} = ds^{(12)}\, x_{12}\, d\varphi$, where γ is surface energy density. The infinitesimal crack length $ds^{(12)}$ is derived as (Rektorys, 1972, 276)

$$ds^{(12)} = dx_{12} \times \sqrt{1 + \left[\frac{\partial f^{(12)}}{\partial x_{12}}\right]^2} \; . \tag{11.4}$$

With regard to the equilibrium condition $dW = dW_{cs}$, we get

$$\frac{\partial f_{12q}}{\partial x_{12}} = \pm \sqrt{\left[\frac{W_{12cq}}{\gamma_q}\right]^2 - 1}, \quad q = M, IN \, . \tag{11.5}$$

where the following condition

$$W_{12cq} - \gamma_q \geq 0, \quad q = M, IN \, . \tag{11.6}$$

is required to be fulfiled. The subscript $q = M$ and $q = IN$ in Equations (11.5), (11.6) is related to the crack formation in the cell matrix and the ellipsoidal inclusion, respectively. Additionally, the condition

$$W_{12cq} - \gamma_q = 0, \quad q = M, IN \, . \tag{11.7}$$

defines a limit state (i.e., a critical state) for the crack formation in the plane x_1x_2 with respect to the parameter $\varphi \in \langle 0, \pi/2 \rangle$, i.e., the infinitesimal crack with the length dx_{12} is formed in the plane x_1x_2 for $\varphi \in \langle 0, \pi/2 \rangle$.

If $f_{12q} = f_{12q}(x_{12}, \varphi)$ is a decreasing or increasing function of the variable x_{12} for the parameter $\varphi \in \langle 0, \pi/2 \rangle$, then the sign "-" or "+" is considered, respectively. With regard to Equation (11.5), if $\partial f_{12q} / \partial x_{12} < 0$ and $\partial W_{12cq} / \partial x_{12} < 0$ or $\partial W_{12cq} / \partial x_{12} > 0$, then $\partial^2 f_{12q} / \partial x_{12}^2 > 0$ or $\partial^2 f_{12q} / \partial x_{12}^2 < 0$, and the decreasing function $f_{12q} = f_{12q}(x_{12}, \varphi)$ of the variable x_{12} for the parameter $\varphi \in \langle 0, \pi/2 \rangle$ is convex or concave is, respectively.

Similarly, if $\partial f_{12q} / \partial x_{12} > 0$ and $\partial W_{12cq} / \partial x_{12} < 0$ or $\partial W_{12cq} / \partial x_{12} > 0$, then $\partial^2 f_{12q} / \partial x_{12}^2 < 0$ or $\partial^2 f_{12q} / \partial x_{12}^2 > 0$, and the increasing function $f_{12q} = f_{12q}(x_{12}, \varphi)$ of the variable x_{12} for the parameter $\varphi \in \langle 0, \pi/2 \rangle$ is concave or convex, respectively.

In case of an intercrystalline crack in polycrystalline materials, we get (Skocovsky and Bokuvka and Palcek, 1996, 93)

$$\gamma_q = \gamma_{Bq}, \quad q = M, IN \, , \tag{11.7}$$

where γ_{Bq} represents inter-atomic bond energy density per unit length, which is related to the boundaries of crystalline grains.

In case of a transcrystalline crack, the energy density per unit length, γ_q, has the form (Brdicka, 2000, 173)

$$\gamma_q = \frac{K_{ICq}}{E_q}, \quad q = M, IN, \tag{11.8}$$

where K_{ICq} and E_q is fracture toughness and Young's modulus, respectively, for the matrix ($q = M$) and the ellipsoidal inclusion ($q = IN$).

The crack formation in the cell matrix and the ellipsoidal inclusion results from the curve integrals W_{12cM} and W_{12cIN}, which are determined with respect to the cubic cell (see Figures 1.2, 11.3). The model system in Figures 1.1 is symmetric. With regard to Figure 11.3, the crack formation in the plane x_1x_2 is sufficient to be investigated within one eighth of the cubic cell, i.e., for the parameter $\varphi \in \langle 0,\pi/2 \rangle$ and the angle $\nu \in \langle 0,\pi/2 \rangle$.

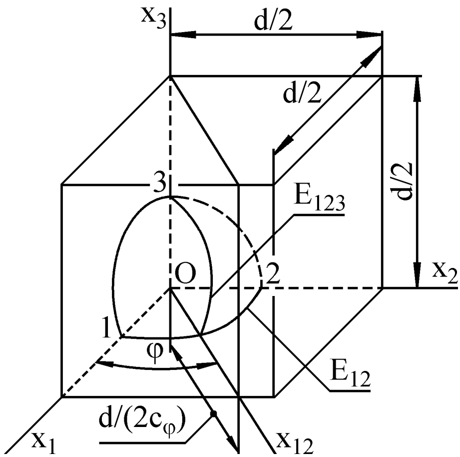


Figure 11.3. One eight of the cubic cell (see Figure 1.2) and the central ellipsoidal inclusion the centre O and with the dimensions $a_{1IN} = O1$, $a_{2IN} = O2$, $a_{3IN} = O3$ along the axes x_1, x_2, x_3 of the Cartesian system ($Ox_1x_2x_3$). The ellipses E_{12}, E_{123} in the planes x_1x_2, $x_{12}x_3$ (see Figures 1.2, 1.4) are given by Equations (1.5), (1.6), respectively, where $\varphi = \angle (x_1,x_{12}) \in \langle 0,\pi/2 \rangle$, $x_{12}x_3 \perp x_1x_2$, $x_{12} \subset x_1x_2$, $x_\varphi \perp x_{12}$. The coefficient c_φ is given by Equation (1.10).

With regard to the plane $x_{12}x_3$ for $\varphi = \angle(x_1, x_{12}) \in \langle 0, \pi/2 \rangle$ in Figure 11.3, the elastic energy density $w_M = w_M(x_n, \varphi, \nu)$ and $w_{IN} = w_{IN}(x_n, \varphi, \nu)$ in the cell matrix (see Equations (5.23), (6.23), (6.34), (6.45), (6.56), (7.22), (7.33), (7.44), (7.55), (8.21), (8.32), (8.43), (8.54), (9.20), (9.31), (9.42), (9.53), and (5.34), (8.63)), respectively, is determined as a function of the coordinates x_n, $\nu \in \langle 0, \pi/2 \rangle$ (see Equations (1.5)-(1.16)).

The elastic energy $w_q = w_q(x_{12}, \varphi, x_3, a_{1IN}, a_{2IN}, a_{3IN}, \nu_{IN})$ ($q = M, IN$) as a function of the cordinates x_{12}, x_3 is determined by the following transformations

$$x_n = \frac{x_3}{\cos\theta}, \quad \sin\nu = \frac{x_{12}}{\sqrt{x_{12}^2 + x_3^2}}, \quad \cos\nu = \frac{x_3}{\sqrt{x_{12}^2 + x_3^2}}, \quad \nu = \arctan\left(\frac{x_{12}}{x_3}\right), \tag{11.9}$$

where $\cos\theta$ is given by Equation (1.12).

11.2 Cell Matrix

The curve integral W_{12cM} of $w_M = w_M(x_{12}, x_3, \varphi, a_1, a_2, a_3, \nu_{IN})$ along the abscissa P_1P_2 (see Figure 11.4) in the plane $x_{12}\,x_3$ of the matrix (see Figure 11.3) is derived as

$$W_{12cM} = W_{12cM}(x_{12}, \varphi, a_{1IN}, a_{2IN}, a_{3IN}, \nu_{IN}) = \int_{P_1P_2} w_M \, dx_3 = \int_0^{d/2} w_M \, dx_3 \,. \tag{11.10}$$

The elastic energy density $w_M = w_M(x_n, \varphi, \nu)$ is a decreasing function of the variable x_n. Consequently, W_{12cM} is a decreasing function of x_{12}, i.e., $\partial W_{12cM} / \partial x_{12} < 0$, and the sign "-" in Equation (11.8) for $q = M$ is considered, i.e., $\partial f_{12cM} / \partial x_{12} < 0$. Due to $\partial W_{12cM} / \partial x_{12} < 0$ and $\partial f_{12cM} / \partial x_{12} < 0$, we get $\partial^2 f_{12cM} / \partial x_{12}^2 > 0$, and the decreasing function $f_{12M} = f_{12M}(x_{12}, \varphi, a_{1IN}, a_{2IN}, a_{3IN}, \nu_{IN})$ of the variable x_{12} for the parameters $\varphi \in \langle 0, \pi/2 \rangle$, a_{1IN}, a_{2IN}, a_{3IN}, ν_{IN} is convex. Consequently, the following condition

$$\left(W_{12cM}\right)_{x_{12}=a_{12}} - \gamma_M = 0. \tag{11.11}$$

represents a transcendental equation with the variable a_{12} and the parameters parameters $\varphi\in\langle 0,\pi/2\rangle$, a_{1IN}, a_{2IN}, a_{3IN}, v_{IN}. The root $x_{12} = a_{12M}^{(X)}$ of this transcendental equation for γ_M given by Equation (11.7) or (11.8) represents such a dimension of the ellipsoidal inclusion along the axis $x_{12} \subset x_1x_2$ (see Figure 11.3), which defines a limit state (i.e., a critical state) for the intercrystalline or transcrystalline matrix crack formation in the plane x_1x_2 with respect to one value of the parameter $\varphi\in\langle 0,\pi/2\rangle$, and then $X = IC$ or $X = TC$, respectively. Accordingly, if $a_{12M}^{(IC)} < a_{12M}^{(TC)}$ or $a_{12M}^{(IC)} > a_{12M}^{(TC)}$, then the intercrystalline or transcrystalline matrix crack is formed in the plane x_1x_2, respectively.

Consequently, if the function $a_{12M}^{(X)} = a_{12M}^{(X)}\left(\varphi, a_{1IN}, a_{2IN}, a_{3IN}\right)$ (X = IC,TC) of the variable $\varphi\in\langle 0,\pi/2\rangle$ exhibits the minimum $a_{\min M}^{(X)}$ for $\varphi = \varphi_{\min M}^{(X)}$, which defines the limit state with respect to the formation of the intercrystalline matrix crack ($X = IC$) or the transcrystalline matrix crack ($X = TC$) in the plane x_1x_2 for each value of the parameter $\varphi\in\langle 0,\pi/2\rangle$ and at the microstructural parameters a_{1IN}, a_{2IN}, a_{3IN}, v_{IN}.

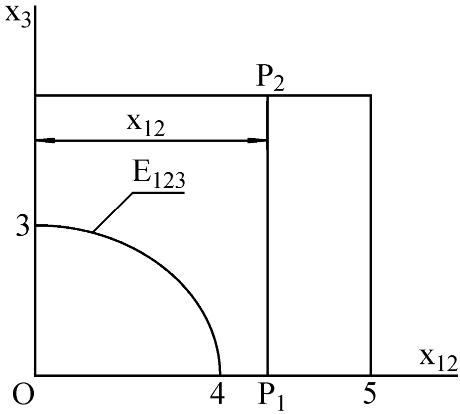


Figure 11.4. The ellipse E_{123} and the abscissa P_1P_2 in the matrix plane x_1x_2 of the cubic cell (see Figure 11.3), where $a_{12} = O4$ and $x_{122} = O5$ are given by Equations (1.6) and (1.9), (1.10), respectively, and $a_3 = O3$.

If $a_{12} > a_{12M}^{(X)}$ $(X = IC,TC)$, then the following condition

$$W_{12cM} - \gamma_M = 0, \quad a_{12} > a_{12M}^{(X)}, \quad X = IC,TC \tag{11.12}$$

represents a transcendental equation with the variable x_{12} and with the root $x_{12} = x_{0M} = x_{0M}(\varphi, a_{1IN}, a_{2IN}, a_{3IN}, v_{IN})$, which defines a position of the crack tip in the matrix (see Figure 11.5).

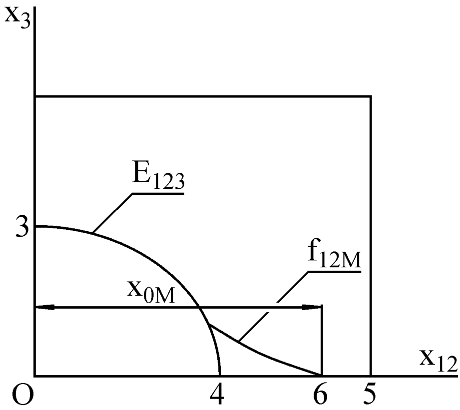


Figure 11.5. The decreasing function $f_{12M} = f_{12M}(x_{12}, \varphi, a_{1IN}, a_{2IN}, a_{3IN}, v_{IN})$ of the variable $x_{12} \in \langle a_{12}, x_{0M} \rangle$, which defines a shape of the matrix crack in the plane $x_{12}x_3$ (see Figure 11.3) for $a_{12} > a_{12M}^{(X)}$ ($X = IC,TC$; see Equations (11.13), (11.14)), where $x_{0M} = x_{0M}(\varphi, a_{1IN}, a_{2IN}, a_{3IN}, v_{IN})$ defines a position of the matrix crack tip, and the microstructural characteristics $a_{1IN}, a_{2IN}, a_{3IN}, v_{IN}$ are parameters of this decreasing function, where $a_{12} = O4$ and $x_{122} = O5$ are given by Equations (1.6) and (1.9), (1.10), respectively, and $a_3 = O3$, $x_{0M} = O6$.

The decreasing function $f_{12M} = f_{12M}(x_{12}, \varphi, a_{1IN}, a_{2IN}, a_{3IN}, v_{IN})$ of the va-riable $x_{12} \in \langle a_{12}, x_{0M} \rangle$ and with the parameters $\varphi \in \langle 0, \pi/2 \rangle$, a_{1IN}, a_{2IN}, a_{3IN}, v_{IN}, which defines a shape of the matrix crack in the plane $x_{12}x_3$ (see Figure 11.3) for $a_{12} > a_{12M}^{(X)}$ $(X = IC,TC)$, has the form

$$f_{12M} = C_M - \int \left(\sqrt{\left[\frac{W_{12cM}}{\gamma_M} \right]^2 - 1} \right) dx_{12}, \quad x_{12} \in \langle a_{12}, x_{0M} \rangle . \tag{11.13}$$

The integration constant $C_M = C_M(\varphi, a_{1IN}, a_{2IN}, a_{3IN}, \nu_{IN})$ is derived as by the condition

$$\left(f_{12M}\right)_{x_{12}=x_{0M}} = 0 , \tag{11.14}$$

and then we get

$$C_M = \left[\int \left(\sqrt{\left[\frac{W_{12cM}}{\gamma_M}\right]^2 - 1} \right) dx_{12} \right]_{x_{12}=x_{0M}} . \tag{11.14}$$

11.3 Ellipsoidal Inclusion

The curve integral W_{12cIN} of $w_{IN} = w_{IN}(x_{12}, x_3, \varphi, a_1, a_2, a_3, \nu_{IN})$ along the abscissa P_1P_2 (see Figure 11.6) in the plane $x_{12}\ x_3$ of the ellipsoidal inclusion (see Figure 11.3) is derived as

$$W_{12cIN} = W_{12cIN}\left(x_{12}, \varphi, a_{1IN}, a_{2IN}, a_{3IN}, \nu_{IN}\right) = \int_{P_1P} w_{IN}\, dx_3 + \int_{PP_2} w_M\, dx_3$$
$$= \int_0^a w_{IN}\, dx_3 + \int_a^{d/2} w_M\, dx_3 , \tag{11.15}$$

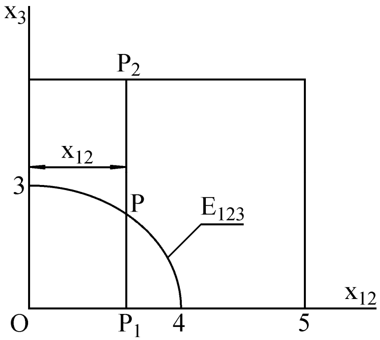


Figure 11.6. The ellipse E_{123} and the abscissa P_1P_2 in the inclusion plane x_1x_2 of the cubic cell (see Figure 11.3), where $a_{12} = O4$ and $x_{122} = O5$ are given by Equations (1.6) and (1.9), (1.10), respectively, and $a_3 = O3$.

where $a_{12} = O4$ and $x_{122} = O5$ are given by Equations (1.6) and (1.9), (1.10), respectively, and then $a = |P_1P|$ is derived as

$$a = \left| P_1 P \right| = \frac{a_3 \sqrt{a_{12}^2 - x_{12}^2}}{a_{12}}, \quad x_{12} \in \langle 0, a_{12} \rangle . \tag{11.16}$$

If W_{12cIN} is a decreasing or increasing function of the variable x_{12}, i.e., $\partial W_{12cIN} / \partial x_{12} < 0$ or $\partial W_{12cIN} / \partial x_{12} > 0$, then the sign "-" or "+" in Equation (11.8) for $q = IN$ is considered, respectively. In both cases we get $\partial^2 f_{12cM} / \partial x_{12}^2 > 0$, and then $f_{12IN} = f_{12IN}(x_{12},\varphi,a_{1IN},a_{2IN},a_{3IN},v_{IN})$ is a convex function of the variable x_{12} for the parameters $\varphi \in \langle 0,\pi/2 \rangle$, a_{1IN}, a_{2IN}, a_{3IN}, v_{IN}.

Consequently, if W_{12cIN} is a decreasing function of x_{12}, then the root $a_{12IN}^{(X)}$ of the following transcendental equation

$$\left(W_{12cIN} \right)_{x_{12}=0} - \gamma_{IN} = 0 . \tag{11.17}$$

represents such a dimension of the ellipsoidal inclusion along the axis $x_{12} \subset x_1x_2$ (see Figure 11.3), which defines a limit state (i.e., a critical state) for the intercrystalline or transcrystalline inclusion crack formation in the plane x_1x_2 with respect to one value of the parameter $\varphi \in \langle 0,\pi/2 \rangle$, and then $X = IC$ or $X = TC$, respectively. Accordingly, if $a_{12IN}^{(IC)} < a_{12IN}^{(TC)}$ or $a_{12IN}^{(IC)} > a_{12IN}^{(TC)}$, then the intercrystalline or transcrystalline inclusion crack is formed in the plane x_1x_2, respectively.

Consequently, if the function $a_{12IN}^{(X)} = a_{12IN}^{(X)}\left(\varphi, a_{1IN}, a_{2IN}, a_{3IN} \right)$ ($X = IC,TC$) of the variable $\varphi \in \langle 0,\pi/2 \rangle$ exhibits the minimum $a_{\min IN}^{(X)}$ for $\varphi = \varphi_{\min IN}^{(X)}$, which defines the limit state with respect to the formation of the intercrystalline inclusion crack ($X = IC$) or the transcrystalline inclusion crack ($X = TC$) in the plane x_1x_2 for each value of the parameter $\varphi \in \langle 0,\pi/2 \rangle$ and at the microstructural parameters a_{1IN}, a_{2IN}, a_{3IN}, v_{IN}.

If $a_{12} > a_{12IN}^{(X)}$ ($X = IC,TC$), then the following conditionIf $a_{12} > a_{12IN}^{(X)}$ ($X = IC,TC$), then the following condition

$$W_{12cIN} - \gamma_{IN} = 0, \quad a_{12} > a_{12IN}^{(X)}, \quad X = IC, TC \tag{11.18}$$

represents a transcendental equation with the variable x_{12} and with the root $x_{12} = x_{0IN} = x_{0IN}(\varphi, a_{1IN}, a_{2IN}, a_{3IN}, \nu_{IN})$, which defines a position of the crack tip in the ellipsoidal inclusion (see Figure 11.6).

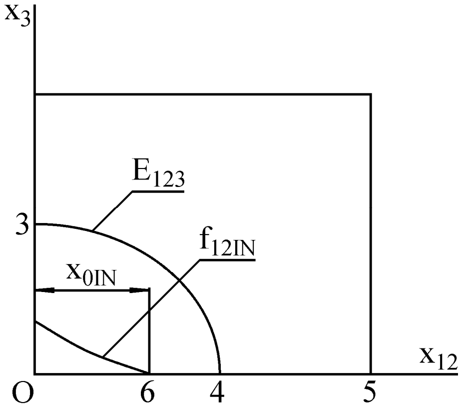


Figure 11.7. The decreasing function $f_{12IN} = f_{12IN}(x_{12}, \varphi, a_{1IN}, a_{2IN}, a_{3IN}, \nu_{IN})$ of the variable $x_{12} \in \langle 0, x_{0IN} \rangle$, which defines a shape of the inclusion crack in the plane $x_{12}x_3$ (see Figure 11.3) for $a_{12} > a_{12IN}^{(X)}$ ($X = IC, TC$; see Equations (11.19), (11.21)), where $x_{0IN} = x_{0IN}(\varphi, a_{1IN}, a_{2IN}, a_{3IN}, \nu_{IN})$ defines a position of the inclusion crack tip, and the microstructural characteristics $a_{1IN}, a_{2IN}, a_{3IN}, \nu_{IN}$ are parameters of this decreasing function, where $a_{12} = O4$ and $x_{122} = O5$ are given by Equations (1.6) and (1.9), (1.10), respectively, and $a_3 = O3$, $x_{0IN} = O6$.

The decreasing function $f_{12IN} = f_{12IN}(x_{12}, \varphi, a_{1IN}, a_{2IN}, a_{3IN}, \nu_{IN})$ of the va-riable $x_{12} \in \langle a_{12}, x_{0IN} \rangle$ and with the parameters $\varphi \in \langle 0, \pi/2 \rangle$, a_{1IN}, a_{2IN}, a_{3IN}, ν_{IN}, which defines a shape of the inclusion crack in the plane $x_{12}x_3$ (see Figure 11.3) for $a_{12} > a_{12IN}^{(X)}$ ($X = IC, TC$), is derived as

$$f_{12IN} = C_{IN} - \int \left(\sqrt{\left[\frac{W_{12cIN}}{\gamma_{IN}} \right]^2 - 1} \right) dx_{12}, \quad x_{12} \in \langle 0, x_{0IN} \rangle . \tag{11.19}$$

The integration constant $C_{IN} = C_{IN}(\varphi, a_{1IN}, a_{2IN}, a_{3IN}, v_{IN})$ is derived as by the condition

$$\left(f_{12IN}\right)_{x_{12}=x_{0IN}} = 0\,, \tag{11.20}$$

and then we get

$$C_{IN} = \left[\int \left(\sqrt{\left[\frac{W_{12cIN}}{\gamma_{IN}}\right]^2 - 1} \right) dx_{12} \right]_{x_{12}=x_{0IN}} \,. \tag{11.21}$$

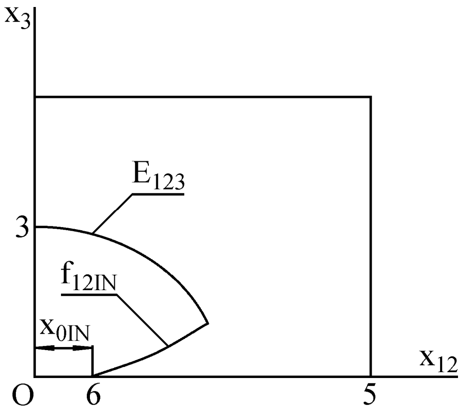


Figure 11.8. The increasing function $f_{12IN} = f_{12IN}(x_{12}, \varphi, a_{1IN}, a_{2IN}, a_{3IN}, v_{IN})$ of the variable $x_{12} \in \langle x_{0IN}, a_{12}\rangle$, which defines a shape of the inclusion crack in the plane $x_{12}x_3$ (see Figure 11.3) for $a_{12} > a_{12IN}^{(X)}$ (X = IC,TC; see Equations (11.19), (11.21)), where $x_{0IN} = x_{0IN}(\varphi, a_{1IN}, a_{2IN}, a_{3IN}, v_{IN})$ defines a position of the inclusion crack tip, and the microstructural characteristics $a_{1IN}, a_{2IN}, a_{3IN}, v_{IN}$ are parameters of this decreasing function, where a_{12} = *O4* and x_{122} = *O5* are given by Equations (1.6) and (1.9), (1.10), respectively, and a_3 = *O3*, x_{0IN} = *O6*.

This analysis is also valid provided that W_{12cIN} is an increasing function of x_{12}, then $x_{12} = a_{12IN}^{(X)}$ represents a root of the following transcendental equation

$$\left(W_{12cIN}\right)_{x_{12}=a_{12}} - \gamma_{IN} = 0\,. \tag{11.22}$$

If $a_{12} > a_{12in}^{(X)}$ $(X = IC,TC)$, then $x_{12} = x_{0IN} = x_{0IN}(\varphi, a_{1IN}, a_{2IN}, a_{3IN}, \nu_{IN})$, which defines a position of the crack tip in the ellipsoidal inclusion (see Figure 11.8), represents a root of the transcendental equation (11.18) with the variable x_{12} and the parameters $\varphi \in \langle 0, \pi/2 \rangle$, a_{1IN}, a_{2IN}, a_{3IN}, ν_{IN}.

The increasing function $f_{12IN} = f_{12IN}(x_{12}, \varphi, a_{1IN}, a_{2IN}, a_{3IN}, \nu_{IN})$ of the va-riable $x_{12} \in \langle x_{0IN}, a_{12} \rangle$ and with the parameters $\varphi \in \langle 0, \pi/2 \rangle$, a_{1IN}, a_{2IN}, a_{3IN}, ν_{IN}, which defines a shape of the inclusion crack in the plane $x_{12}x_3$ (see Figure 11.3) for $a_{12} > a_{12IN}^{(X)}$ $(X = IC,TC)$, is derived as

$$f_{12IN} = \int \left(\sqrt{\left[\frac{W_{12cIN}}{\gamma_{IN}} \right]^2 - 1} \right) dx_{12} - C_{IN}, \quad x_{12} \in \langle x_{0IN}, a_{12} \rangle . \quad (11.23)$$

The integration constant $C_{IN} = C_{IN}(\varphi, a_{1IN}, a_{2IN}, a_{3IN}, \nu_{IN})$ is derived as by the condition (11.20), and then we get

$$C_{IN} = \left[\int \left(\sqrt{\left[\frac{W_{12cIN}}{\gamma_{IN}} \right]^2 - 1} \right) dx_{12} \right]_{x_{12} = x_{0IN}} . \quad (11.24)$$

CHAPTER 12

APPENDIX

12.1 Wronskian's method

Wronskian's method can be explained by the following mathematical example. The differential equation (6.3) with a non-zero right side is derived as

$$\frac{\partial^2 u_n}{\partial x_n^2} + \frac{2}{x_n}\frac{\partial u_n}{\partial x_n} - \frac{2u_n}{x_n^2} = g, \quad g = \frac{C_1}{x_n} + C_2\, x_n^{c3-2} + \frac{C_3}{x_n^2}\,, \tag{12.12}$$

where the integration constants C_1, C_1, C_1, are determined by the boundary condition (4.1)-(4.5). If $g = 0$, then we get

$$x_n^2 \frac{\partial^2 u_n}{\partial x_n^2} + 2x_n \frac{\partial u_n}{\partial x_n} - 2u_n = 0\,. \tag{12.13}$$

If $u_n = x_n^{\lambda}$, then the solutions u_{1n}, u_{2n} have the forms

$$u_{1n} = x_n, \quad u_{2n} = \frac{1}{x_n^2}\,. \tag{12.14}$$

The solution u_n of Equation (10.13) is derived as (Rektorys, 1973, 341-345)

$$u_n = a_1 u_{1n} + a_2 u_{2n}, \quad a_i = \int \frac{W_2^{(i)}}{W_2}\, dx_n, \quad i = 1,2\,. \tag{12.15}$$

Wronskian's determinants W_2, $W_2^{(i)}$ (i = 1,2) have the forms

$$W_2 = \begin{vmatrix} u_{1n}, & u_{2n} \\ \dfrac{\partial u_{1n}}{\partial x_n}, & \dfrac{\partial u_{1n}}{\partial x_n} \end{vmatrix}, \quad W_2^{(1)} = \begin{vmatrix} 0, & u_{2n} \\ g, & \dfrac{\partial u_{1n}}{\partial x_n} \end{vmatrix}, \quad W_2^{(2)} = \begin{vmatrix} u_{1n}, & 0 \\ \dfrac{\partial u_{1n}}{\partial x_n}, & g \end{vmatrix}, \tag{12.16}$$

where the determinant $W_2^{(i)}$ $(i = 1,2)$ is created from W_2, i.e., the i-th column of W_2 is replaced by the following one

$$\left.\begin{matrix} 0 \\ g \end{matrix}\right\} 2 \ rows \ . \tag{12.17}$$

Let $f_1, \ldots, f_n$ represent n solutions of a differential equation of the n-th rand with zero right-hand side (i.e., $g = 0$). Let the functions $f_1, \ldots, f_n$ of the variable x exhibit contionous derivatives to the (n-1)-th degree. The solution of this differential equation with a non-zero right-hand side (i.e., $g \neq 0$) is derived as

$$f = \sum_{i=1}^{n} a_i \, f_i, \quad a_i = \int \frac{W_n^{(i)}}{W_n} dx \, . \tag{12.18}$$

Wronskian's determinant W_n with n rows and n columns has the form

$$W_n = \begin{vmatrix} f_1, & f_2, & \ldots, & f_n \\ \dfrac{\partial f_1}{\partial x}, & \dfrac{\partial f_2}{\partial x}, & \ldots, & \dfrac{\partial f_n}{\partial x} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial^{n-1} f_1}{\partial x^{n-1}}, & \dfrac{\partial^{n-1} f_2}{\partial x^{n-1}}, & \ldots, & \dfrac{\partial^{n-1} f_n}{\partial x^{n-1}} \end{vmatrix}, \tag{12.19}$$

where $W_n^{(i)}$ $(i = 1,\ldots,n)$ is created from W_n, i.e., the i-th column of W_n is replaced by the following one

$$\left.\begin{matrix} 0 \\ 0 \\ \vdots \\ g \end{matrix}\right\} n \ \ rows \ . \tag{12.20}$$

12.2 Cramer's rule

The system of *n* linear algehraic equations is derived as

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{1n}x_n = b_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n \end{matrix} , \tag{12.1}$$

where the root x_i (i = 1,…,n) is determined by Cramer's rule (Rektorys, 1973, 22-28)

$$x_i = \frac{D_n^{(i)}}{D_n}, \quad i = 1,...,n \ , \tag{12.2}$$

The determinant D_n with *n* rows and *n* columns has the form

$$D_n = \begin{vmatrix} a_{11}, & a_{12}, & ... , & a_{1n} \\ a_{21}, & a_{22}, & ... , & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1}, & a_{n2}, & ... , & a_{nn} \end{vmatrix}$$

$$= \sum_{i=1}^{n} (-1)^{1+i} a_{1i} \, D_{n-1}^{(1i)} = \sum_{i=1}^{n} (-1)^{1+i} a_{i1} \, D_{n-1}^{(i1)} \ , \tag{12.3}$$

and the subdeterminant $D_n^{(i)}$ is created from D_n, i.e., the i-th column of D_n is replaced by

$$\left.\begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{matrix}\right\} n \ \ rows \ . \tag{12.4}$$

Similarly, the subdeterminant $D_{n-1}^{(ij)}$ with (n-1) rows and (n-1) columns is created from D_n, i.e., the i-th row and the j-column are omitted. If n = 2, we get

$$D_2 = \begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix} = a_{11}\,a_{22} - a_{12}\,a_{21}\,. \tag{12.5}$$

Similarly, if n = 3, we get

$$D_3 = \begin{vmatrix} a_{11}, & a_{12}, & a_{13} \\ a_{21}, & a_{22}, & a_{23} \\ a_{31}, & a_{32}, & a_{33} \end{vmatrix}$$

$$= a_{11} \begin{vmatrix} a_{22}, & a_{23} \\ a_{32}, & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21}, & a_{23} \\ a_{31}, & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21}, & a_{22} \\ a_{31}, & a_{32} \end{vmatrix} . \tag{12.6}$$

12.3 Integrals

The derivatives of the functions $f = x^{\lambda}$, $f = \ln x$ and the constant C have the forms (Rektorys, 1973, 234-247)

$$\left(x^{\lambda}\right)' = \lambda\,x^{\lambda-1}, \quad \left(\ln x\right)' = \frac{1}{x}, \quad C' = 0\,. \tag{12.7}$$

The indefinite integrals of $f = x^{\lambda}$ and $f = C$ are derived as

$$\int x^{\lambda}\,dx = \frac{x^{\lambda+1}}{\lambda+1}, \quad \lambda \neq 1; \quad \int C\,dx = C\,x\,. \tag{12.8}$$

In case of the product $f\ g$ of the function $f = f(x)$, $g = g(x)$, we get (Rektorys, 1973, 234-247)

$$(fg)' = f'g + fg' , \tag{12.9}$$

and then the integral of $f\,g$ is derived as

$$\int f\,g\,dx = f\,g - \int f\,g'\,dx \ . \tag{12.10}$$

With regard to Equations (6.13)-(6.15), (7.13), (7.14), the following integrals have the forms

$$\int x^{\lambda} \ln x\,dx = \frac{x^{\lambda+1}}{\lambda+1}\ln x - \int \frac{x^{\lambda+1}}{\lambda+1}\times\frac{1}{x}dx = \frac{x^{\lambda+1}}{\lambda+1}\ln x - \frac{1}{\lambda+1}\int x^{\lambda}\ dx$$
$$= \frac{x^{\lambda+1}}{\lambda+1}\left(\ln x - \frac{1}{\lambda+1}\right), \quad \lambda \neq 1,$$
$$\int \ln x\,dx = \int 1\times\ln x\,dx = x\ln x - \int x\times\frac{1}{x}\,dx = x\ln x - \int 1\times dx =$$
$$= x\ln x - \int 1\times dx = x(\ln x - 1) ,$$
$$\int x^{\lambda}\ln^2 x\,dx = \frac{1}{\lambda+1}\left(x^{\lambda+1}\ln^2 x - \int x^{\lambda}\ln x\,dx\right)$$
$$= \frac{x^{\lambda+1}}{\lambda+1}\left[\left(\ln x - \frac{1}{\lambda+1}\right)^2 + \frac{1}{(\lambda+1)^2}\right], \quad \lambda \neq 1 . \tag{12.11}$$

12.4 Numerical determination

Numerical values of the thermal and phase-transformation stresses for real matrix-inclusion composites include integrals and derivatives, which are determined by a programming language. If $f = f(\mathrm{x})$, then the numerical value of the derivative $\partial f/\partial x$ is determined by

$$\frac{\partial f}{\partial x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} . \tag{12.21}$$

In case of the angles φ, ν (see Figure 1.4), the step $\Delta x = \Delta\varphi = \Delta\nu = 10^{-6}$ [deg] is sufficient.

Let F represent a definite integral of the function $f = f(\varphi, \nu)$ with the variables $\varphi, \nu \in \langle 0,\ \pi/2 \rangle$. Let n, m be integral parts of the real numbers $\pi/(2\Delta\varphi)$, $\pi/(2\Delta\nu)$, respectively. Numerical values of the definite integral F are determined by the following formula

$$F = \int_0^{\pi/2} \int_0^{\pi/2} f(\varphi,\nu)\,d\varphi\,d\nu \approx \sum_{j=0}^{m} \left(\sum_{i=0}^{n} f(i\times\Delta\varphi; j\times\Delta\nu)\Delta\varphi \right)\Delta\nu\,, \quad (12.22)$$

where the step $\Delta\varphi = \Delta\nu = 0.1$ [deg] is sufficient. The average numerical value $\overline{f}$ of the function $f = f(\varphi, \nu)$ with the variables $\varphi, \nu \in \langle 0,\ \pi/2 \rangle$ is determined by the following formula

$$\overline{f} = \left(\frac{2}{\pi}\right)^2 \int_0^{\pi/2} \int_0^{\pi/2} f(\varphi,\nu)\,d\varphi\,d\nu \approx \left(\frac{2}{\pi}\right)^2 \sum_{j=0}^{m} \left(\sum_{i=0}^{n} f(i\times\Delta\varphi; j\times\Delta\nu)\Delta\varphi \right)\Delta\nu \quad (12.23)$$

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